MA103 - Class 5

1 Let R be an equivalence relation on a set S. Prove that the following properties hold.

(a) For all $x, y \in S$ we have $xRy \iff [x] = [y]$.

(b) For all $x, y \in S$ we have $\neg xRy \iff [x] \cap [y] = \emptyset$.

a) The statement is: $\forall x,y \in S, \ (xRy \Rightarrow [x]=[y])$

not $(\forall x,y \in S, \ xRy) \Rightarrow [x]=[y]$, which has no meaning.

"$\Rightarrow$" Let $x,y \in S$ arbitrary such that xRy. Since R is transitive, $\forall a \in S, \ aRx \Rightarrow aRy$ (since by hypothesis xRy).
Therefore, let $z \in [x]$. By definition of $[x]$, we have zRx, and therefore zRy. This implies that $\forall a \in [x], \ a \in [y]$.

"$\Leftarrow$" Let $x,y \in S$ such that $[x]=[y]$. Since R is reflexive, we have xRx and therefore $x \in [x]=[y]$. By definition, this means xRy.

b) How to prove an implication by contradiction: Say we want to prove $P \Rightarrow Q$ by contradiction. Then we assume that P AND $\neg Q$ hold, and we find a contradiction.
If you want to prove by contradiction that $\forall x \in A, \ P(x) \Rightarrow Q(x)$ you assume $\exists x \in A, \ P(x) \wedge \neg Q(x)$.

"$\Rightarrow$" Assume by contradiction that there exist $x,y \in S$ with $\neg xRy$ and $[x] \cap [y] \neq \emptyset$. Let $z \in [x] \cap [y]$ (which we can do since $[x] \cap [y] \neq \emptyset$. By definition, xRz and yRz. Since R is symmetric and transitive, this gives us xRy.

"$\Leftarrow$" Assume by contradiction there exist x,y such that $\neg(\neg xRy)$ and $[x] \cap [y] = \emptyset$. We know that $[x]=[y]$ by the previous exercise. Moreover, $[x] \neq \emptyset$ since R is reflexive and therefore $x \in [x]$.

2 In lectures, we gave a construction for the rational numbers. This started by looking at the set S of all pairs of the form (a,b), with $a,b \in \mathbb{Z}$ and $b \neq 0$, and then considering the relation Q on $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ defined by:

$$(a,b)Q(c,d) \qquad \text{if and only if} \qquad ad = bc.$$

(a) Explain why things would go badly wrong if we allow S to include pairs (a,b) with $b = 0$.

(Hint: the answer does not involve 'division by zero'.)

We defined the set $\mathbb{Q}$ to be the set of equivalence classes of the relation Q, and defined an "addition" operation on $\mathbb{Q}$ by setting $[(a,b)] \oplus [(e,f)] = [(af+be, bf)]$, for each $(a,b),(e,f) \in S$.

- You <u>CANNOT</u> get a contradiction from a definition.

The idea is to show that one of the needed properties does not work.

$(3,2)\, R\, (0,0)$; $(0,0)\, R\, (1,1)$

but $\neg (3,2)\, R\, (1,1)$. (calculations needed).

Now we "define" an operation. We just saw that we cannot just "define" stuff. We need to check that everything works properly.

e.g. $f(x) = \begin{cases} x^2 & \text{if } x^2 - 2 \geq 0 \\ x^2+1 & \text{if } x^3 - 5x^2 + \sqrt{3}x \geq 0 \end{cases}$

what <u>could</u> go wrong in this example? What do we need to check? What is the equivalent result in our problem?

We need to check that if $(a,b)\, Q\, (a',b')$ and $(c,d)\, Q\, (c',d')$ then $[(a,b)] \oplus [(c,d)] = [(a',b')] \oplus [(c',d')]$.

(b) Suppose that (a,b), (c,d) and (e,f) are in S, and that $(a,b)Q(c,d)$. Show that $(af+be, bf)Q(cf+de, df)$.

Your answer should *only* talk about operations with integers. If your answer involves writing any fractions at any stage, it is wrong.

Use this to show that if $(r,s)Q(t,u)$ and $(v,w)Q(x,y)$, then

$$[(r,s)] \oplus [(v,w)] = [(t,u)] \oplus [(x,y)]\,.$$

(This means that the addition operation defined on $\mathbb{Q}$ is *well-defined*.)

we know $ad = bc$, and we want to prove $(af+be)df = (cf+de)bf$

CLASS SLIDES for WEEK 6.