MA103 - Class 7

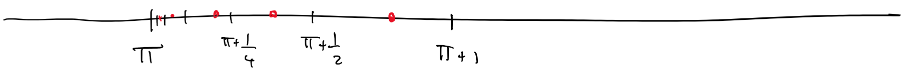
GENERAL REMARKS
- good assignments
- last few weeks

TOPOLOGICAL/GEOMETRICAL APPROACH TO $\mathbb{R}$

1. $\mathbb{N}$ we count
2. $\mathbb{Z}$ we do "operations" with $+, -$
3. $\mathbb{Q}$ we do "operations" with $\cdot, \div$
4. $\mathbb{R}$ roots of polynomials?? Not really.

There is an "intermediate" step between $\mathbb{Q}$ and $\mathbb{R}$: the set of ALGEBRAIC NUMBERS $\mathcal{A}$.
The main thing that works NICELY in $\mathbb{R}$ but not in $\mathbb{Q}$ or $\mathcal{A}$ are SEQUENCES.

Take a_n s.t. $\forall n \in \mathbb{N}$, $a_n \in \left(\pi + \frac{1}{2^n}, \pi + \frac{1}{2^{n-1}}\right) \cap \mathbb{Q}$



This is a bounded decreasing sequence in $\mathbb{Q}$ (therefore in $\mathcal{A}$) that has no limit in $\mathcal{A}$ (therefore in $\mathbb{Q}$).

Q Can you prove it? Can you prove that $\forall q \in \mathbb{Q}$, the sequence does not converge to q?

DEF (CONVERGENT SEQUENCE) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and L in $\mathbb{R}$. We say that (a_n) converges to L if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N \quad |a_n - L| < \varepsilon$$

EXE 2.4 Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence with limit L.
If $\forall n \in \mathbb{N}$ $a_n \geq 0$, then $L \geq 0$.

Rem It is not true that $\forall n \in \mathbb{N}$ $a_n > 0 \Rightarrow L > 0$.
Can you find a counterexample?

$\rightsquigarrow$ PROOF

NEW EXERCISE Let $S \subseteq \mathbb{R}$ be a non-empty, upper bounded set of real numbers. Then there exists a sequence $(a_n)_{\mathbb{N}}$ in S (with all the elements in S) such that

$$\lim_{n\to\infty} a_n = \sup(S).$$

TRY IT BEFORE reading the hints.

⓪ What happens if S is $\{x\}$?
— " — FINITE?
— " — $(0,1)$?
— " — an open interval (a,b)?

① Let $u^* := \sup(S)$. Then:
- $\forall x \in S,\ x \leq u^*$
- u^* is the least upper bound

② You need to find a sequence similar to the examples!

BE CAREFUL. Does your system work for

$$[0,1] \setminus \{1 - \tfrac{1}{n} : n \in \mathbb{N}\}?$$

This is non-empty and upper bounded.