MA210 - Class 3

QUESTION 3 Let $a_m :=$ # m-digit $\{0,1\}$-sequences s.t. no two consecutive 0s are allowed

b) Show that for $m \geq 3$, $a_m = a_{m-1} + a_{m-2}$

$\rightsquigarrow$ Let $\underline{x} = x_1, \dots, x_k$ be a $\{0,1\}$-sequence. We say that $P(\underline{x})$ holds if there are no two consecutive 0s. Let us also denote with $\{0,1\}^m$ the set of ALL $\{0,1\}$-seq. of length m.

$$\forall m \in \mathbb{N}, \qquad A_m := \{\underline{x} \in \{0,1\}^m \mid P(\underline{x}) \text{ holds}\}.$$

CLAIM Let $\underline{x} = x_1, \dots, x_k$ be a sequence s.t. $P(\underline{x})$ holds. Say $k \geq 3$. If $\underline{y}$ is obtained from $\underline{x}$ by deleting some initial digits, then $P(\underline{y})$ holds.

$\rightsquigarrow$ If there are no two consecutive 0s in $\underline{x}$, we cannot find them in $\underline{y}$.

Let $m \geq 3$.

Rem $$A_m = \underbrace{\{\underline{x} \in A_m \mid x_1 = 1\}}_{B_m} \sqcup \underbrace{\{\underline{x} \in A_m \mid x_1 = 0\}}_{C_m}$$

Now consider that $\forall \underline{x} \in C_m$, $x_2 = 1$, otherwise $\underline{x} \notin A_m$ bc $\neg P(\underline{x})$. Consider

$$f_B : B_m \longrightarrow A_{m-1} \qquad f_C : C_m \longrightarrow A_{m-2}$$
$$\underline{x} \longmapsto (x_2, \dots, x_m) \qquad \underline{x} \longmapsto (x_3, \dots, x_m)$$

- by CLAIM, these are well defined

- $$g_B : A_{m-1} \longrightarrow B_m \qquad g_C : A_{m-2} \longrightarrow C_m$$
$$\underline{y} \longmapsto (1, y_1, \dots, y_{m-1}) \qquad \underline{y} \longmapsto (0, 1, y_1, \dots, y_{m-2})$$

are proper inverses of f_B and f_C, which therefore have to be bijections.

$$\left(f_B \circ g_B = Id_{A_{m-1}} \ , \ g_B \circ f_B = Id_{B_m} \ , \ldots\ldots \right)$$

QUESTION 6 $\quad a_m := \#$ of m-digits $\{-1,0,1\}$ strings s.t. no consecutive 0s or 1s are allowed.

a) Show $\forall m \geq 3, \quad a_m = 2a_{m-1} + a_{m-2}$

$\leadsto$ Define $\{-1,0,1\}^m := \{$ seq. of length m in $0,1,-1\}$

for $\underline{x} \in \{-1,0,1\}^n$, let $P(\underline{x})$ hold iff $\underline{x}$ has no consecutive 0s or 1s.

$$A_m := \{ \underline{x} \in \{-1,0,1\}^m \mid P(\underline{x}) \text{ holds} \}.$$

For $i \in \{1,-1,0\}$ let $\quad A_m^i := \{ \underline{x} \in A_m \mid x_1 = i \}$.

CLAIM Let $\underline{x} = x_1, \ldots, x_k$ be a sequence s.t. $P(\underline{x})$ holds. Say $k \geq 3$. If $\underline{y}$ is obtained from $\underline{x}$ by deleting some initial digits, then $P(\underline{y})$ holds.

$\leadsto$ If there are no two consecutive 0s or 1s in $\underline{x}$, we cannot find them in $\underline{y}$.

Rem $\forall m \geq 3$ we have

$$A_m = A_m^{-1} \sqcup A_m^0 \sqcup A_m^1$$

CLAIM ① $|A_m^{-1}| = |A_{m-1}|$

② $|A_m^0| = |A_{m-1}^{-1}| + |A_{m-1}^1|$

③ $|A_m^1| = |A_{m-1}^0| + |A_{m-1}^{-1}|$

$\leadsto$ ②+③. Let $\underline{x} \in A_m^0$ (resp. A_m^1 up to switch digits accordingly). Then by CLAIM, $P(x_2, \ldots, x_m)$ holds. Moreover $x_2 \in \{1,-1\}$ otherwise $P(\underline{x})$ would not hold.
Therefore $\underline{x} \in A_{m-1}^{-1} \cup A_{m-1}^1$ which are clearly disjoint.

① $f: \check{A}_m^{-1} \longrightarrow A_{m-1}$ has an inverse
$\quad \underline{x} \longmapsto (x_2, \dots, x_m)$

which is the function $g: A_{m-1} \longrightarrow A_m^{-1}$
$\quad \underline{x} \longmapsto (-1, x_1, \dots, x_{m-1})$

up to checking it is an inverse, we are done ■

Conclusion: $|A_m| = |A_m^{-1}| + |A_m^0| + |A_m^1|$

$$= |A_{m-1}| + \underbrace{|A_{m-1}^{-1}| + |A_{m-1}^1| + |A_{m-1}^0|}_{|A_{m-1}|} + \underbrace{|A_{m-1}^{-1}|}_{|A_{m-2}^{-1}|}$$

$$= 2|A_{m-1}| + |A_{m-2}|$$

Question 2 SUMMER 2019

(a) Let a_n be the number of solutions, for any r, of $x_1 + x_2 + \cdots + x_r = n$ such that each term x_i is either 1 or 2. In other words, it is the number of ways to write n as an ordered sum of 1's and 2's. For example,

$a_1 = 1$: The only solution is $1 = 1$ $(x_1 = 1, r = 1)$.

$a_2 = 2$: The two solutions are $1 + 1 = 2$ $(x_1 = x_2 = 1, r = 2)$, and $2 = 2$ $(x_1 = 2, r = 1)$.

$a_3 = 3$: The three solutions are $1 + 1 + 1 = 3$ $(x_1 = x_2 = x_3 = 1, r = 3)$, $1 + 2 = 3$ $(x_1 = 1, x_2 = 2, r = 2)$, and $2 + 1 = 3$ $(x_1 = 2, x_2 = 1, r = 2)$.

(i) Prove that $a_4 = 5$.

(ii) Prove that the sequence satisfies the recurrence relation

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2.$$

i) CASE A: no 2s. Then just 1 possibility: 1+1+1+1

CASE B: one 2. 2+1+1 ; 1+2+1 ; 1+1+2 are possible

case C: two 2s. 2+2 is the only possibility

ii) Let $A_m := \{x \in \bigcup_{i \in \mathbb{N}} \{1,2\}^i \mid \sum x_j = m\}$.

Fix $m \geq 3$ (it is already proved for $m = 2$).

$B_m := \{x \in A_m \mid x_1 = 1\}$ $\qquad C_m := \{x \in A_m \mid x_1 = 2\}$.

For $x \in \{1,2\}^i$, let $\bar{x} := (x_2, \dots, x_i)$, which is let it denote the same seq. but truncated.

Consider $f_{\_}: B_m \longrightarrow A_{m-1}$ $\qquad f_c: C_m \longrightarrow A_{m-2}$

$$B \quad x \longmapsto \bar{x} \qquad\qquad x \longmapsto \bar{x}$$

These functions have left and right inverses, which are

$$g_B : A_{n-1} \longrightarrow B_n \qquad\qquad g_C : A_{n-2} \longrightarrow C_n$$
$$x \longmapsto (1, x_1, \dots) \qquad\qquad x \longmapsto (2, x_1, \dots)$$

Therefore we have

$$|A_n| = |B_n| + |C_n| = |A_{n-1}| + |A_{n-2}|$$

$\blacksquare$

(b) Find the generating function of the sequence given by the recurrence relation

$$b_n = b_{n-1} + b_{n-2} \text{ for } n \geq 2, \quad b_0 = 2, \quad b_1 = 1.$$

b) We have to use the def. of gen. fun:

$$f(x) := \sum_{n=0}^{\infty} b_n x^n = 2 + x + \sum_{n=2}^{\infty} b_n x^n$$

$$\overset{\text{recur. rel.}}{=} 2 + x + \sum_{n=2}^{\infty} (b_{n-1} + b_{n-2}) x^n$$

$$= 2 + x + \sum_{n=2}^{\infty} b_{n-1} x^n + \sum_{n=2}^{\infty} b_{n-2} x^n$$

$$= 2 + x + x \sum_{n=1}^{\infty} b_n x^n + x^2 \sum_{n=0} b_n x^n$$

$$= 2 + x + x\left(f(x) - 2\right) + x^2 f(x)$$

$$\leadsto \quad -2 + x = (x^2 - x - 1) f(x)$$

which is, $f(x) = \dfrac{-2 + x}{x^2 + x - 1}$