MA210 - Class 4

17. (a) Suppose we roll a six-sided die (English: one die, many dice... even if not all native speakers know this). Let d_n be the number of possible ways to roll a die so that the outcome is n.

Explain why the generating function of the sequence $d_0, d_1, \ldots$ is given by

$$f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6.$$

(b) Suppose that we roll four (distinguishable) dice. Let a_n be the number of throws such that the sum of outcomes is equal to n.

Explain why the generating function of the sequence $a_0, a_1, \ldots$ is given by

$$g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

(c) Now let b_n be the number of throws with any number of (distinguishable) dice such that the sum of outcomes is equal to n.

Explain why the generating function of the sequence $b_0, b_1, \ldots$ is given by

$$h(x) = \sum_{n=0}^{\infty} (x + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

Prove that $h(x) = (1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}$.

a) If we roll a die just once, we have at most one way to get to any number. Moreover we have a way to get to $1, 2, 3, \ldots, 6$. So

$$d_m = \begin{cases} 1 & \text{if } m = 1, \ldots, 6 \\ 0 & \text{otherwise.} \end{cases}$$

By definition, the generating function is $x + x^2 + \ldots + x^6$

b) let $d_m^{(i)}$:= # ways to throw i dice such that the sum is equal to m.

CONSIDER $d_{\ldots}^{(2)}$ FIRST

$d_m^{(2)}$ = # ways to get tot of m with two dice

$d_m^{(2)}$ when possible:

die 1	die 2
0	m
1	$m-1$
$\vdots$	$\vdots$
m	0

Since die 1 can get to the number k in $d_k^{(1)}$ ways, we have

$$\boxed{d_m^{(2)} = \sum_{k=0} d_k^{(1)} \cdot d_{m-k}^{(1)}}$$

By definition, this means that $d_m^{(2)}$ is the convolution of $d_m^{(1)}$ and $d_m^{(1)}$.

Moreover, we have the gen. fun. of $d_m^{(1)}$ is $f(x) = x + x^2 + \dots + x^6$. By CONVOLUTION THM we get that the gen. fun. of $d_m^{(2)}$ is $f(x) \cdot f(x)$.

Consider now $d_m^{(3)}$

dice 1,2	die 3
m	0
$m-1$	1
	2
$\vdots$	$\vdots$
0	m

So we get $d_m^{(3)} = \sum_{k=0}^{m} d_k^{(2)} d_{m-k}^{(1)}$

INDUCTION We can take $i \geq 4$, and consider. We have

dice $1,2,\dots,i-1$	die i
m	0
$m-1$	1
	2
$\vdots$	$\vdots$
0	m

therefore

$$d_m^{(i)} = \sum_{k=0}^{m} d_k^{(i-1)} d_{m-k}^{(1)}$$

We can once again use convolution theorem.

c) $b_m = d_m^{(1)} + d_m^{(2)} + d_m^{(3)} + \dots = \sum_{i=1}^{\infty} d_m^{(i)}$. We have

that the gen. fun. of b_m is the sum of the gen. functions of $d_m^{(i)}$. which means

$$h(x) = \sum_{k=0}^{\infty} f^{(k)}(x) = \sum_{k=0}^{\infty} (x + \dots + x^6)^k = \frac{1}{1 - x - x^2 - \dots - x^6}$$

18. The British coin system has 1p, 2p, 5p, 10p, 20p, 50p, £1 = 100p, and £2 = 200p coins.

(a) Let a_n count the number of different ways that you can pay a sum of n pennies with 10p coins.

Find the generating function of the sequence $a_0, a_1, \dots$.

(b) Let b_n count the number of different ways that you can pay a sum of n pennies with 1p and 2p coins.

Find the generating function of the sequence $b_0, b_1, \dots$ and determine the closed formula for b_n.

(c) Let c_n count the number of different ways that you can pay a sum of n pennies

(c) Let c_n count the number of different ways that you can pay a sum of n pennies.

Find the generating function of the sequence $c_0, c_1, \ldots$.

a) We have a situation similar to the dice situation earlier.

$$a_m = \begin{cases} 1 & m \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

So $f(x) = 1 + x^{10} + x^{20} + x^{30} + \ldots = \dfrac{1}{1 - x^{10}}$

Let $a_m^{(k)}$ be the sequence that counts the way of paying using only coins of value k.

b) We have $a_m^{(1,2)} =$

# pennies paid with 1 penny	# pennies paid with 2 pennies
m	0
$m-1$	1
$m-2$	2
$\vdots$	$\vdots$
0	m

So we have $a_m^{(1,2)} = \sum_{k=0}^{m} a_k^{(1)} a_{m-k}^{(2)}$. So by conv. thm. we get $f^{(1,2)}(x) = \dfrac{1}{1-x} \cdot \dfrac{1}{1-x^2}$

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$$a_m^{(1,2)} = \begin{cases} \frac{1}{2}m + 1 & m \text{ even} \\ \frac{1}{2}(m+1) & m \text{ odd} \end{cases}$$

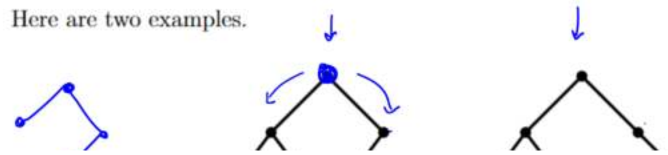
c) Similar to previous exercise.

A *binary tree* is something that is important in computer science (and algorithms in general).

This is an object that you can create as follows. First, draw a dot (which we call a *vertex* on the paper. Then apply the following procedure as often as you like.

Pick a vertex with no lines coming down from it, and draw either a line (which we call an *edge*) going down to a new vertex on the left, or going down to a new vertex on the right, or both. We call the first vertex the *root*. Each vertex other than the root has an edge to exactly one vertex above it, its *parent*, and zero, one or two vertices below it, its *children*.

Here are two examples.



These two binary trees are different — the root's right child has only a left child in the first example, and only a right child in the the second.

Let C_n be the number of n-vertex binary trees. It's convenient to define $C_0 = 1$.

We want to find the generating function for C_n. We have

$$C_n = \sum_{i=0}^{n-1} C_i \, C_{n-i-1} \qquad n > 0$$

But CONV. THM. is in the form: $a_n = \sum_{k=0}^{n} b_k + c_{n-k}$. So we can define $d_n := c_{n+1}$. Now we have $d_n = c_{n+1} = \sum_{i=0}^{n} c_i \cdot c_{n-i}$ so d_n is the convolution of c_n and c_n.

So let f be the gen. fun. of c_n. We get by conv. then that $f(x) \cdot f(x)$ is the gen. fun. of d_n.

$$\begin{aligned} f^2(x) &= \sum_{n=0}^{\infty} d_n x^n = \sum_{n=0}^{\infty} c_n x^n \\ &= \frac{1}{x} \sum_{n=0}^{\infty} c_{n+1} x^{n+1} \\ &= \frac{1}{x} \sum_{n=1}^{\infty} c_n x^n = \frac{1}{x}\left(\sum_{x=0}^{\infty} c_n x^n - 1\right) \\ &= \frac{1}{x}(f(x) - 1). \end{aligned}$$

So we get $f^2(x) = \frac{1}{x}(f(x)-1)$. Solving by $f(x)$ yields (for $x \neq 0$) $f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$

But $f(0) = C_0 = 1$ and generating functions should be continuous at 0.

So we want $\lim_{x \to 0} f(x) = f(0) = 1$. So $f(x) = \frac{1-\sqrt{1-4x}}{2x}$

We can use this formula to get C_n.

Hint: use the series for $\sqrt{1+y}$.

You should get $C_n = \frac{1}{n+1}\binom{2n}{n}$