MA423 - Class 5

Wednesday, 27 October 2021 11:36

Def Let S be a discrete (finite) set. An (S-valued, discrete time) STOCHASTIC PROCESS is a sequence $X_0, X_1, \dots$ of random variables with values in S.
Such a stoch. process is called a MARKOV CHAIN if $\forall t \in \mathbb{N}, \forall x_0, x_1, \dots, x_{t+1} \in S$ we have

$$\mathbb{P}[X_{t+1} = x_{t+1} \mid X_0 = x_0, \dots, X_t = x_t] = \mathbb{P}[X_{t+1} = x_{t+1} \mid X_t = x_t].$$

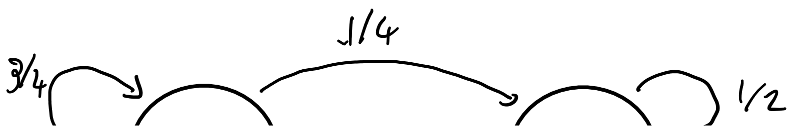
1. **(Submit, 4pts)** A consumer who purchases one of two brands of soap powder every week is influenced by her choice of the previous week but not by earlier experience. If she purchased brand A the previous week, her current purchase would be of the same brand with probability $3/4$ and brand B with probability $1/4$. If she purchased brand B the previous week, the probability she would again purchase brand B is $1/2$, and the probability of purchasing A is also $1/2$.

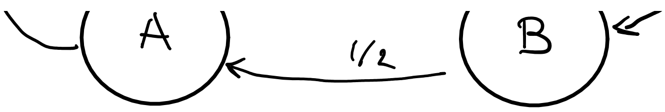
   (a) Model this stochastic process as a time-homogeneous Markov chain, where the brand purchased in Week t is the state of the process at time t. Write the transition matrix P of the Markov chain.

   (b) Compute P^2. Interpret the values of P^2.

   (c) Given that we use soap A in Week 0, what is the probability that our sequence of purchases from Week 0 to Week 4 is A, B, A, A, B

a)





For all states $i, j \in \{0, 1, \ldots, M-1\}$, the *transition probability* from state i to state j is

$$p_{ij} = P(X_{t+1} = j | X_t = i)$$

All of these probabilities can be presented in an $M \times M$ matrix P whose (i,j)-entry, that is, the entry in row i and column j, is equal to p_{ij}. This matrix is called the *transition matrix.*

Therefore we can write

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}$$

b) Computing P^2 is an algebraic problem.

$$P^2 = \begin{pmatrix} 0.6875 & 0.3215 \\ 0.625 & 0.375 \end{pmatrix}$$

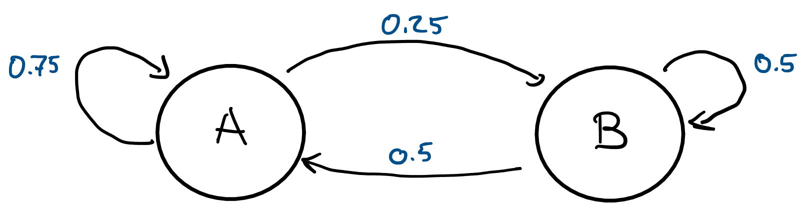
What does it mean?

THM CHAPMAN-KOLMOGOROV! $\qquad P^{(m)} = P^m$

Where $P^{(m)}$ is the m-th step transition probability matrix.

which is, $P^{(m)}_{i,j} = \mathbb{P}[X_m = j \mid X_0 = i]$.

c) Given that we use soap A in week 0, what is the probability that the purchase sequence is ABAAB?



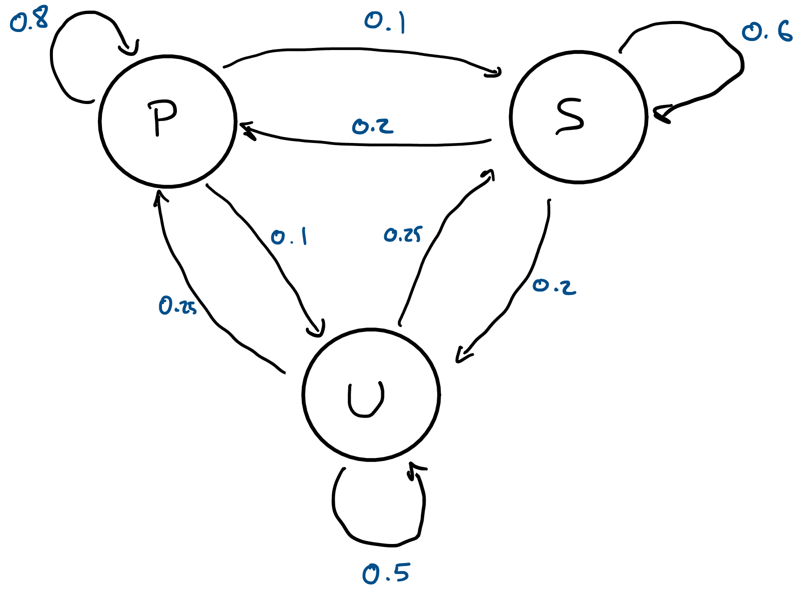
Remember This is a TIME HOMOGENEOUS Markov chain.

Therefore $\mathbb{P}[X_{t+1} = j \mid X_t = i] = \mathbb{P}[X_1 = j \mid X_0 = i] = P_{ij}$

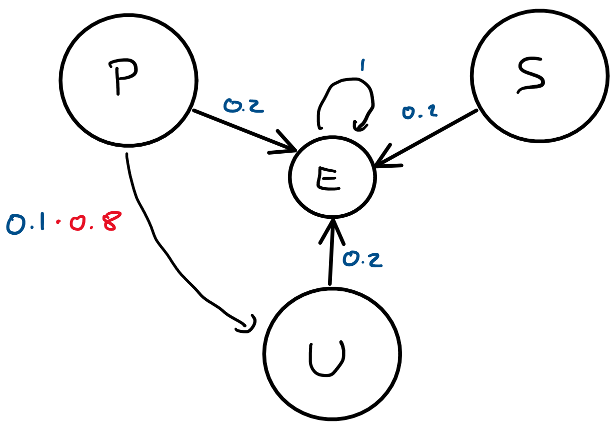
So we need to compute $P_{AB} \cdot P_{BA} \cdot P_{AA} \cdot P_{AB} = 0.02343$

PROBLEM 2 Problem about modelling a Markov chain.

We obtain the following graph

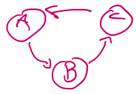


c) We want to add a 0.2 chanche that the line estinguishes, how do we model this?

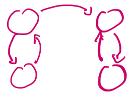


THEORETICAL INTERMEZZO

IRREDUCIBLE Markov chain: if every pair of states communicate with each other.

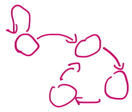


CLASS of STATES is a maximal set of mutually communicating states.

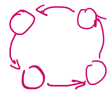


⟶ absorbing states are classes

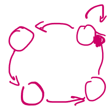
TRANSIENT / RECURRENT state   A state A is transient if there exists a state B such that for some n, $P^n_{AB} > 0$ but $\forall i$, $P^i_{BA} = 0$.
Otherwise, A is recurrent.



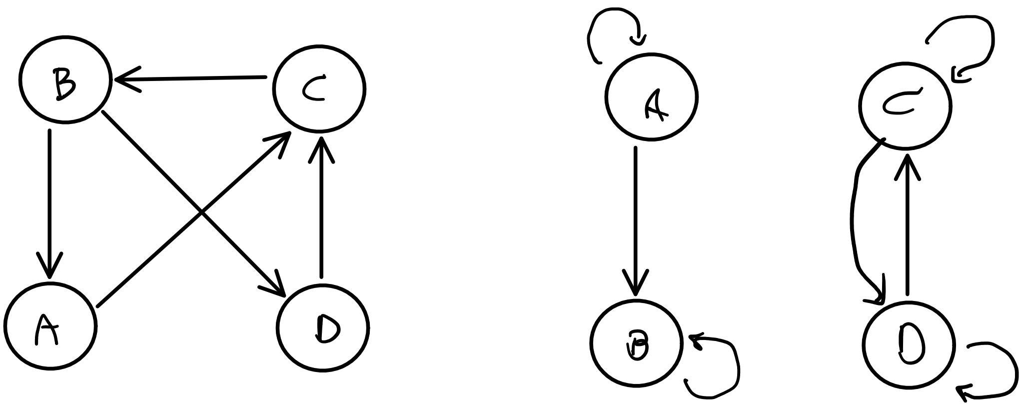
PERIOD of a state A is the gcd of all the i for which $P^i_{AA} > 0$.
If the period is 1, the state is APERIODIC



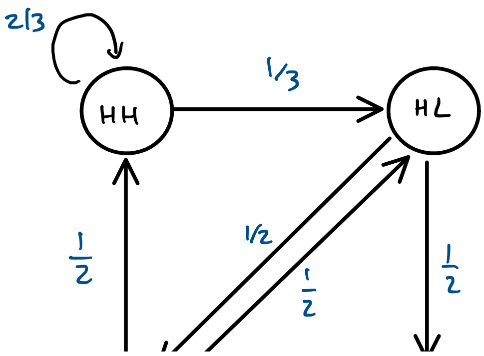
ERGODIC = APERIODIC + RECURRENT



PROBLEM 3   Find the classes, transient/recurrent, absorbing, period.



PROBLEM 4



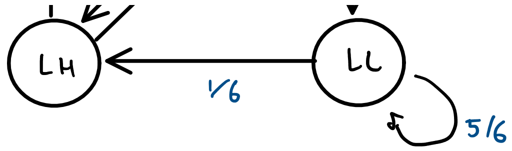
Theorem 9.1 (Steady State Probabilities). *For every irreducible ergodic Markov chain, there exists non-zero probabilities $\pi_j, j = 0, 1, \ldots, M-1$ for the states, called the* steady state probabilities, *such that starting from an arbitrary state i,*

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$$

holds. Moreover, the π_j's are the unique solutions to the steady state equations:

$$\pi_j = \sum_{i=0}^{M-1} \pi_i p_{ij} \quad \text{for all } j = 0, 1, \ldots, M-1$$

$$\sum_{j=0}^{M-1} \pi_j = 1$$



Compute the STEADY STATE PROBABILITIES

$$\begin{cases} \pi_{HH} = \frac{2}{3}\pi_{HH} + \frac{1}{2}\pi_{LH} \\ \pi_{HL} = \frac{1}{3}\pi_{HH} + \frac{1}{2}\pi_{LH} \\ \pi_{LH} = \frac{1}{2}\pi_{HL} + \frac{1}{6}\pi_{LL} \\ \pi_{LL} = \frac{1}{2}\pi_{HL} + \frac{5}{6}\pi_{LL} \\ 1 = \pi_{HH} + \pi_{LH} + \pi_{HL} + \pi_{LL} \end{cases}$$

$\pi_{HH} = \frac{3}{13}$

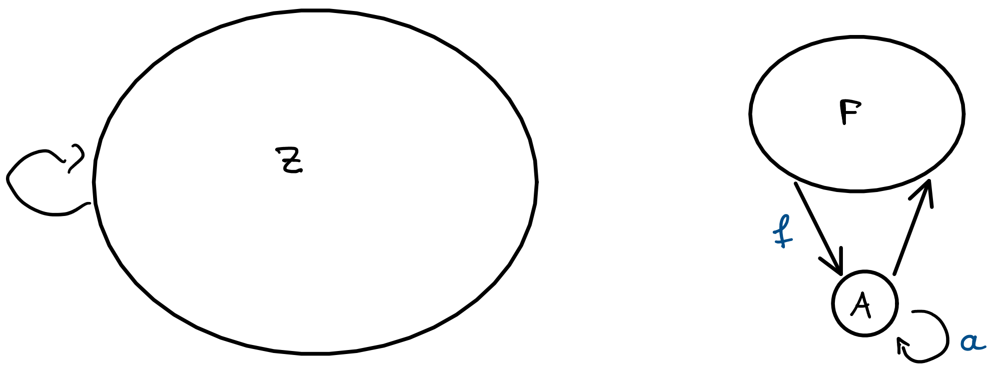
$\pi_{HL} = \pi_{LH} = \frac{2}{13}$

$\pi_{LL} = \frac{6}{13}$

NEW EXERCISE

A group of LSE students tries to promote their website *www.ORrocks.co.uk* by hacking the PageRank algorithm. It is 1998 and the Internet currently consists of 990 pages only. Besides their new website, they created 9 fake sites to promote it (thus the total number of pages reaching 1000). PageRank is computed using random teleportation with $d = 0.2$. Assume the students cannot influence the old pages that do not link to the 10 new ones; however, they can add links arbitrarily among the new sites. Advise them on how to add links in order to maximise the PageRank of *www.ORrocks.co.uk*. What is the maximum value they can reach? Sites are also allowed to link themselves.

(1) We can treat each "group" of states as a unique state.



(2) By taking into account the 0.2 teleportation ...

② By taking into account the 0.2 repopulation factor we get:

$$\left\{\begin{aligned} Z &= \frac{1}{5}\,\frac{990}{1000} + \frac{4}{5}\,Z \\ F &= \frac{1}{5}\,\frac{9}{1000} + \frac{4}{5}\left(A - a + F - f\right) \\ A &= \frac{1}{5}\,\frac{1}{1000} + \frac{4}{5}\left(a + f\right) \end{aligned}\right\}$$

$$\left\{\begin{aligned} Z &= 95/100 \\ A + F &= 1/100 \end{aligned}\right\}$$

$$\left\{\begin{aligned} A &= 41/5000 \\ F &= 9/5000 \end{aligned}\right.$$