ETH zürich

Minimal Ramsey Graphs for cyclicity

A paper by D. Reding and A. Taraz ['18]

Speaker: Domenico Mergoni
Supervisor: Charlotte Knierim

Contents of this seminar:

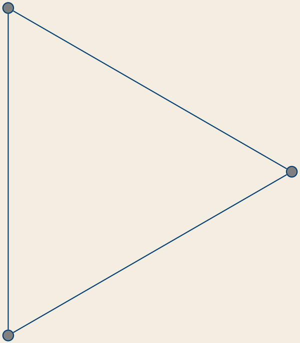
r-Ramsey graphs

Def: r-Ramsey graph

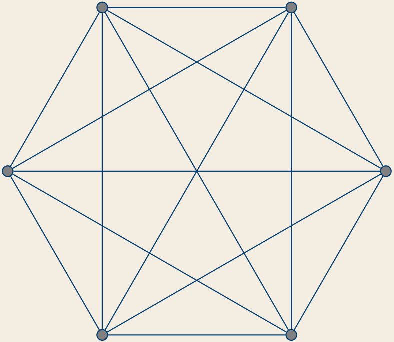
Let H be a graph and $r \in \mathbb{N}_{\geq 2}$, an ***r*-Ramsey graph** for H is a graph G such that any r-edge-colouring of G admits a monochromatic copy of H. The set of such graphs will be denoted by $\mathscr{R}_r(H)$.

2-Ramsey graph for a triangle

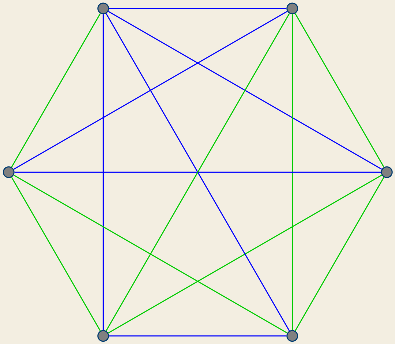
Take H a triangle and $r = 2$



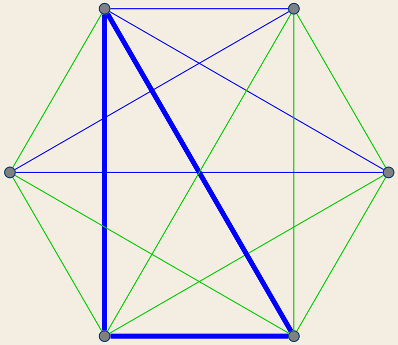
2-Ramsey graph for a triangle



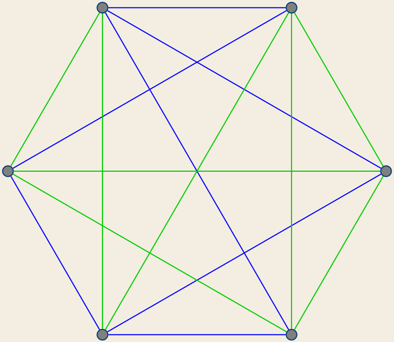
2-Ramsey graph for a triangle



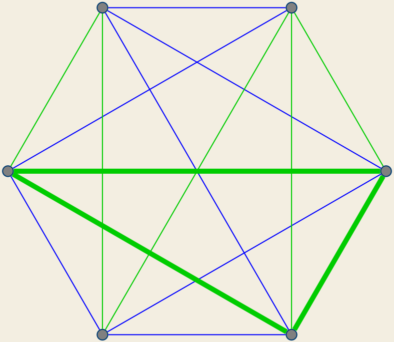
2-Ramsey graph for a triangle



2-Ramsey graph for a triangle



2-Ramsey graph for a triangle



r-Ramsey graphs

Def: r-Ramsey graph

Let H be a graph and $r \in \mathbb{N}_{\geq 2}$, an **$r$-Ramsey graph** for H is a graph G such that any r-edge-colouring of G admits a monochromatic copy of H. The set of such graphs will be denoted by $\mathcal{R}_r(H)$.

Example: $\mathcal{R}_r(K_{1,m})$

It is clear that we have $K_{1,r\cdot m} \in \mathcal{R}_r(K_{1,m})$.

Minimal r-Ramsey graphs

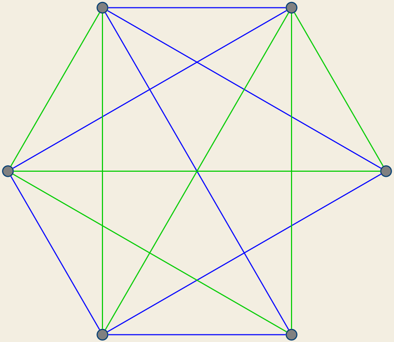
Def: r-Ramsey graph

Let H be a graph and $r \in \mathbb{N}_{\geq 2}$, an **$r$-Ramsey graph** for H is a graph G such that any r-edge-colouring of G admits a monochromatic copy of H. The set of such graphs will be denoted by $\mathscr{R}_r(H)$.

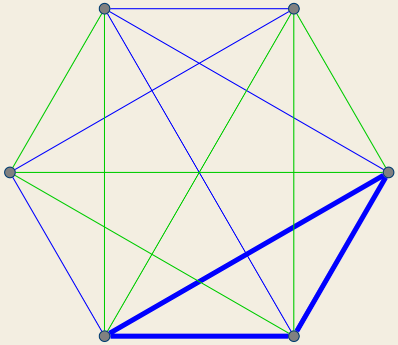
Def: Minimal r-Ramsey graph

$G \in \mathscr{R}_r(H)$ is called a minimal r-Ramsey graph if every proper subgraph of G is not in $\mathscr{R}_r(H)$. We write $G \in \mathscr{M}_r(H)$.

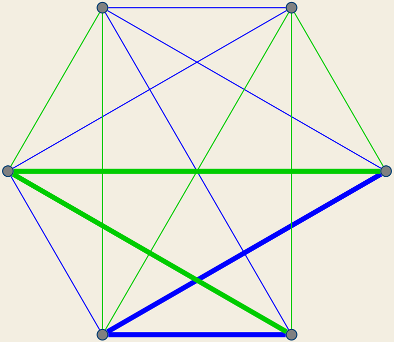
Minimal r-Ramsey graphs



Minimal r-Ramsey graphs



Minimal r-Ramsey graphs



Minimal r-Ramsey graphs

Def: r-Ramsey graph

Let H be a graph and $r \in \mathbb{N}_{\geq 2}$, an **$r$-Ramsey graph** for H is a graph G such that any r-edge-colouring of G admits a monochromatic copy of H. The set of such graphs will be denoted by $\mathcal{R}_r(H)$.

Def: Minimal r-Ramsey graph

$G \in \mathcal{R}_r(H)$ is called a minimal r-Ramsey graph if every proper subgraph of G is not in $\mathcal{R}_r(H)$. We write $G \in \mathcal{M}_r(H)$.

Example: $\mathcal{R}_2(K_{1,m})$

It is clear that we have $K_{1,2m-1} \in \mathcal{M}_2(K_{1,m})$.

Minimal r-Ramsey graphs

Ramsey's theorem

Fix $r, k \in \mathbb{N}_{\geq 2}$, we can find monochromatic k-cliques in any r-edge colouring of a sufficiently large complete graph.

Minimal r-Ramsey graphs

Ramsey's theorem

Fix $r, k \in \mathbb{N}_{\geq 2}$, we can find monochromatic k-cliques in any r-edge colouring of a sufficiently large complete graph.

Note

Ramsey's theorem implies that $\mathscr{R}_r(H)$ and $\mathscr{M}_r(H)$ are never empty. And explain the general interest towards the study of $\mathscr{R}_r(K_n)$.

Minimal r-Ramsey graphs

Ramsey's theorem

Fix $r, k \in \mathbb{N}_{\geq 2}$, we can find monochromatic k-cliques in any r-edge colouring of a sufficiently large complete graph.

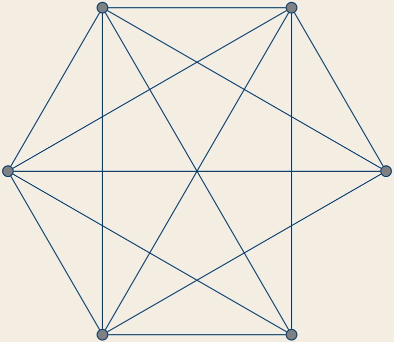
Note

Ramsey's theorem implies that $\mathcal{R}_r(H)$ and $\mathcal{M}_r(H)$ are never empty. And explain the general interest towards the study of $\mathcal{R}_r(K_n)$.

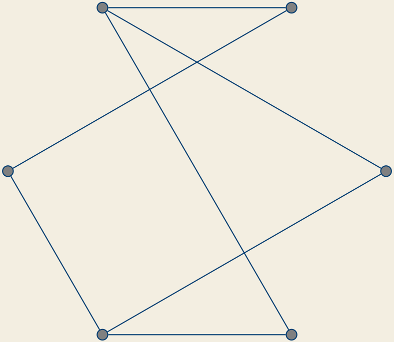
Main problem in the study of such objects

Describing how a graph G can be edge-decomposed in H-free graphs.

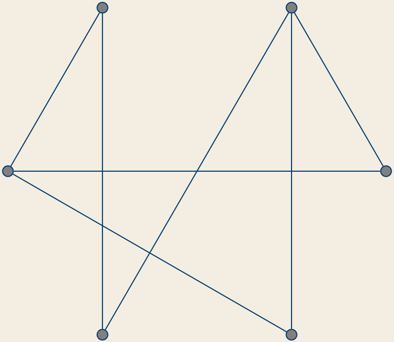
Minimal r-Ramsey graphs



Minimal r-Ramsey graphs



Minimal r-Ramsey graphs



What do we know about this kind of objects?

Theorem by Clemens, Liebenau, Reding [2018]

If H is 3-connected or isomorphic to a triangle, then:

- for $2 \leq r \leq q$ if $G \in \mathscr{M}_r(H)$ then G is an induced subgraph for infinitely many $F \in \mathscr{M}_q(H)$.
- for $3 \leq r$ there are $G \in \mathscr{M}_r(H)$ of arbitrarily large maximum degree, genus, and chromatic number.

What do we know about this kind of objects?

Theorem by Clemens, Liebenau, Reding [2018]

If H is 3-connected or isomorphic to a triangle, then:

- for $2 \leq r \leq q$ if $G \in \mathcal{M}_r(H)$ then G is an induced subgraph for infinitely many $F \in \mathcal{M}_q(H)$.
- for $3 \leq r$ there are $G \in \mathcal{M}_r(H)$ of arbitrarily large maximum degree, genus, and chromatic number.

Theorem by Rödl, Siggers [2008]

Exponential lower bound in n for the number of non-isomorphic graphs on $\mathcal{M}_r(K_k)$ on at most n vertices.

r-Ramsey graphs for a property $\mathscr{P}$

Def: r-Ramsey graph for a property $\mathscr{P}$

Let $\mathscr{P}$ be a property, closed under taking supergraphs and $r \in \mathbb{N}_{\geq 2}$. Then an **$r$-Ramsey graph for $\mathscr{P}$** is a graph G such that any r edge - colouring of G admits a monochromatic copy of a member of $\mathscr{P}$.

r-Ramsey graphs for a property $\mathscr{P}$

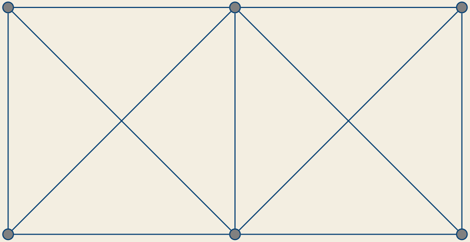
Def: r-Ramsey graph for a property $\mathscr{P}$

Let $\mathscr{P}$ be a property, closed under taking supergraphs and $r \in \mathbb{N}_{\geq 2}$. Then an **$r$-Ramsey graph for $\mathscr{P}$** is a graph G such that any r edge - colouring of G admits a monochromatic copy of a member of $\mathscr{P}$.

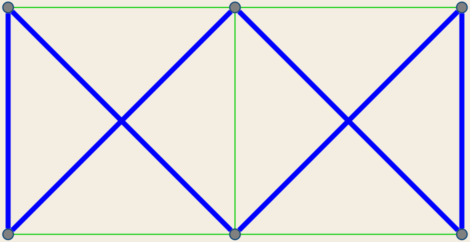
Examples

This class has been studied for the properties of: connectivity (W. Mader ['72]), minimum degree (R.Klein, J. Schönheim ['61]), planarity (J.Battle, F. Haray ['62]) and **ciclicity**.

r-Ramsey graphs for ciclicity



r-Ramsey graphs for ciclicity



r-Ramsey graphs for a property $\mathscr{P}$

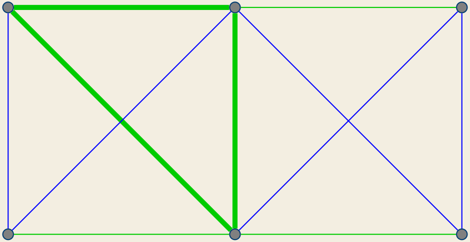
Def: r-Ramsey graph for a property $\mathscr{P}$

Let $\mathscr{P}$ be a property, closed under taking supergraphs and $r \in \mathbb{N}_{\geq 2}$. Then an **$r$-Ramsey graph for $\mathscr{P}$** is a graph G such that any r edge - colouring of G admits a monochromatic copy of a member of $\mathscr{P}$.

Note

The choice of such a member can depend on the colouring.

Minimal *r*-Ramsey graphs for ciclicity



r-Ramsey graphs for a property $\mathscr{P}$

Def: r-Ramsey graph for a property $\mathscr{P}$

Let $\mathscr{P}$ be a property, closed under taking supergraphs and $r \in \mathbb{N}_{\geq 2}$. Then an **$r$-Ramsey graph for $\mathscr{P}$** is a graph G such that any r edge - colouring of G admits a monochromatic copy of a member of $\mathscr{P}$.

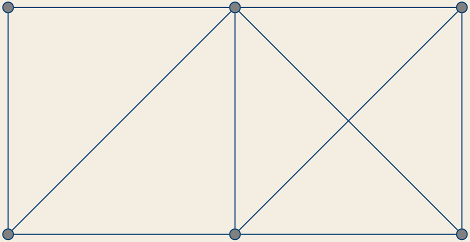
Note

The choice of such a member can depend on the colouring.

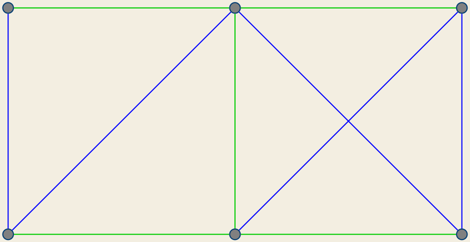
Def: Minimal r-Ramsey graph for $\mathscr{P}$

$G \in \mathscr{R}_r(\mathscr{P})$ is called a minimal r-Ramsey graph for $\mathscr{P}$ if every proper subgraph of G is not in $\mathscr{R}_r(\mathscr{P})$. We write $G \in \mathscr{M}_r(\mathscr{P})$.

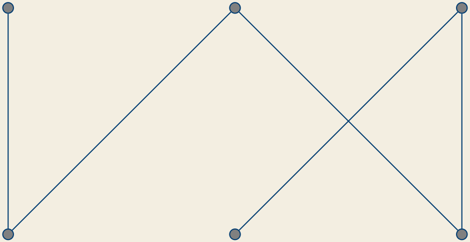
Minimal r-Ramsey graphs for ciclicity



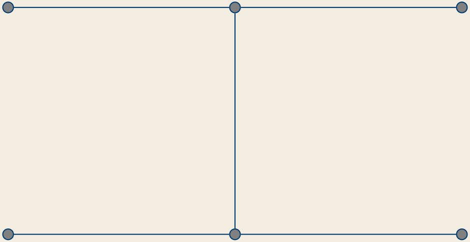
Minimal *r*-Ramsey graphs for ciclicity



Minimal r-Ramsey graphs for ciclicity



Minimal *r*-Ramsey graphs for ciclicity



One example: The class $\mathscr{R}_r(\mathscr{C}_{odd})$

Lemma

Let $r \geq 2$, then $G \in \mathscr{R}_r(\mathscr{C}_{odd})$ if and only if $\chi(G) \geq 2^r + 1$.

One example: The class $\mathscr{R}_r(\mathscr{C}_{odd})$

Lemma

Let $r \geq 2$, then $G \in \mathscr{R}_r(\mathscr{C}_{odd})$ if and only if $\chi(G) \geq 2^r + 1$.

Proof (*Idea*)

- $\leftarrow$ If $G \notin \mathscr{R}_r(\mathscr{C}_{odd})$, then G edge-decomposes into $\leq r$ bipartite graphs. Therefore $\chi(G) \leq 2^r$.
- $\rightarrow$ Induction on r.

One example: The class $\mathscr{R}_r(\mathscr{C}_{odd})$

Lemma

Let $r \geq 2$, then $G \in \mathscr{R}_r(\mathscr{C}_{odd})$ if and only if $\chi(G) \geq 2^r + 1$.

Proof (*Idea*)

- $\leftarrow$ If $G \notin \mathscr{R}_r(\mathscr{C}_{odd})$, then G edge-decomposes into $\leq r$ bipartite graphs. Therefore $\chi(G) \leq 2^r$.
- $\rightarrow$ Induction on r.

Idea

Study minimal graphs with $\chi(G) = (2^r + 1)$ instead of $\mathscr{M}_r(\mathscr{C}_{odd})$.

Characterization of Ramsey graphs for cyclicity

Nash-Williams arboricity theorem [1964]

Every graph G admits an edge-decomposition into $\lceil ar(G) \rceil$ many forests and none with less forests. Where we define

$$ar(G) := \max_{J \subseteq G, v_J > 1} \frac{e_j}{v_j - 1}$$

Characterization of Ramsey graphs for cyclicity

Nash-Williams arboricity theorem [1964]

Every graph G admits an edge-decomposition into $\lceil ar(G) \rceil$ many forests and none with less forests. Where we define

$$ar(G) := \max_{J \subseteq G, v_J > 1} \frac{e_j}{v_j - 1}$$

Corollary

$G \in \mathscr{R}_r(\mathscr{C})$ iff $\frac{e_H}{v_H - 1} > r$ for some subgraph $H \subseteq G$. Because $G \in \mathscr{R}_r(\mathscr{C})$ iff G do not edge-decompose in r forests.

Characterization of Ramsey graphs for cyclicity

Nash-Williams arboricity theorem [1964]

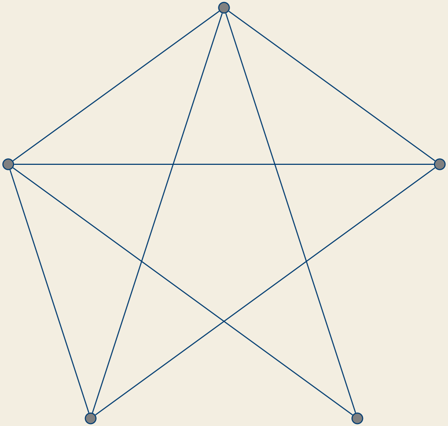
Every graph G admits an edge-decomposition into $\lceil ar(G) \rceil$ many forests and none with less forests. Where we define

$$ar(G) := \max_{J \subseteq G, v_J > 1} \frac{e_J}{v_J - 1}$$

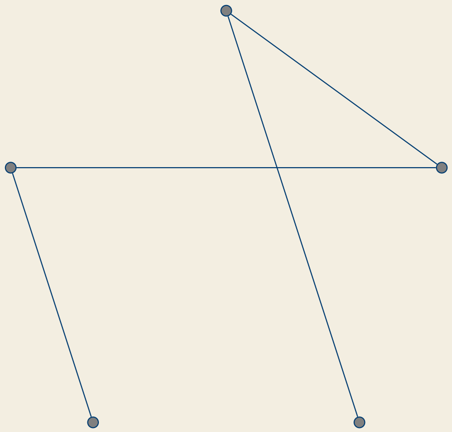
Corollary

In particular $G \in \mathscr{M}_r(\mathscr{C})$ iff $\frac{e_H}{v_H - 1} < r, \ \forall H \subsetneq G$ and $\frac{e_G}{v_G - 1} \geq r$.

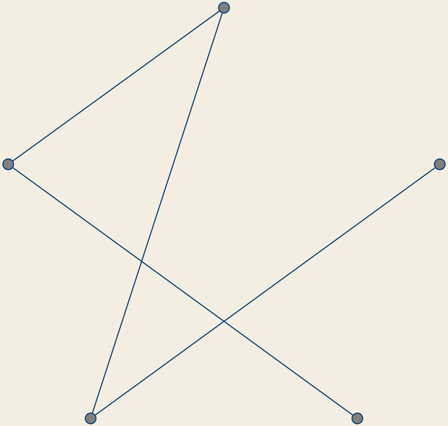
Characterization of Ramsey graphs for cyclicity



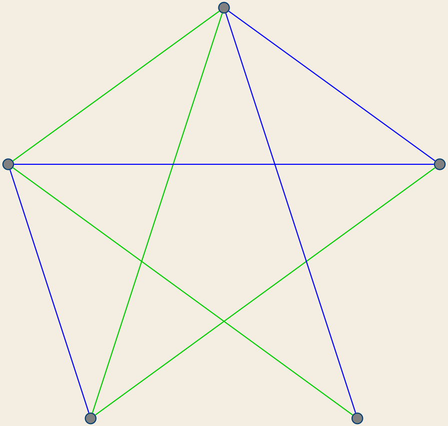
Characterization of Ramsey graphs for cyclicity



Characterization of Ramsey graphs for cyclicity



Characterization of Ramsey graphs for cyclicity



First aim of the paper

Note

From now on we will assume $r = 2$.

Idea

We mentioned the relation between $\mathscr{M}(\mathscr{C})$ and minimal graphs with $\chi(G) = 5$. These were completely described by Hàjos [1961] as the graphs that could be minor-constructed from K_5.
We want to do the same with $\mathscr{M}(\mathscr{C})$.

Construction of $\mathscr{M}(\mathscr{C})$

Theorem I

For every $G \in \mathscr{M}(\mathscr{C})$ there exists a finite sequence of minimal Ramsey graphs for $\mathscr{C}$ such that (if $\prec$ denotes the minor relation):

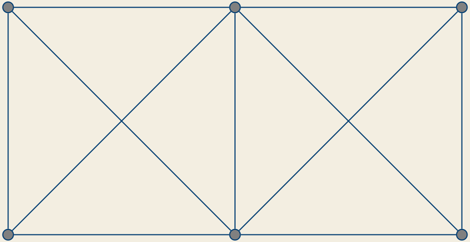
$$\{K_5 - e, K_4 \vee K_4\} \ni G_0 \prec \cdots \prec G_n = G$$

Idea

This means that every $G \in \mathscr{M}(\mathscr{C})$ can be obtained starting from one base graph and recursively splitting one suitable vertex. We will see how in the second theorem of the paper.

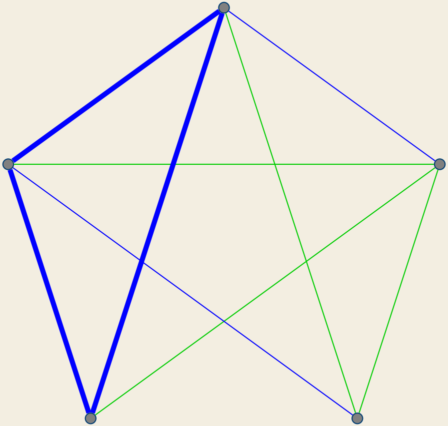
Base graphs

$K_4 \vee K_4$



Base graphs

$K_5 - e$



Proof of theorem I

Lemmas

Lemma I.1

Every $G \in \mathscr{M}(\mathscr{C})$ is 2-connected and satisfies $\delta(G) = 3$.

Proof of theorem I

Lemmas

Lemma I.1

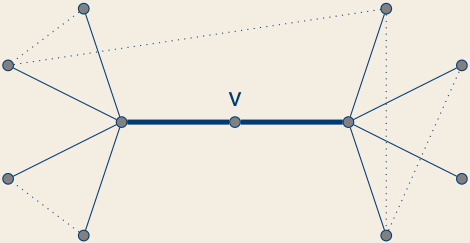
Every $G \in \mathscr{M}(\mathscr{C})$ is 2-connected and satisfies $\delta(G) = 3$.

Idea

- Upper bound for $\delta(G)$ is given by the characterization of $\mathscr{M}(\mathscr{C})$ which follows from arboricity theorem.
- Lower bound for $\delta(G)$: suppose $d(v) = 2$, then $G \setminus v \in \mathscr{R}(\mathscr{C})$.

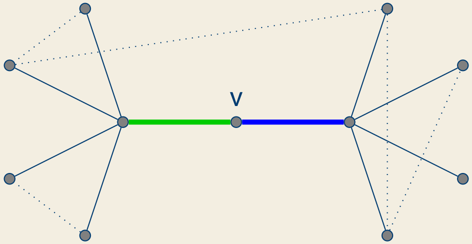
Proof of theorem I

$d(v) > 2$



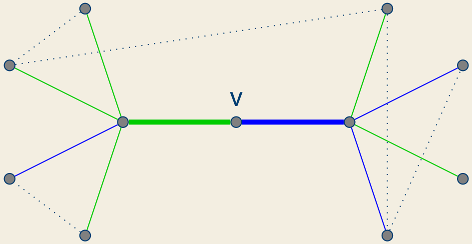
Proof of theorem I

$d(v) > 2$



Proof of theorem I

$d(v) > 2$



Proof of theorem I

Lemmas

Lemma I.1

Every $G \in \mathcal{M}(\mathcal{C})$ is 2-connected and satisfies $\delta(G) = 3$.

Idea

- Upper bound for $\delta(G)$ is given by the characterization of $\mathcal{M}(\mathcal{C})$ which follows from arboricity theorem.
- Lower bound for $\delta(G)$: suppose $d(v) = 2$, then $G \setminus v \in \mathcal{R}(\mathcal{C})$.
- Suppose $G = G_1 \cup G_2$ s.t. $G_1 \cap G_2$ has at most one vertex. Fix a colouring of G_2 s.t. it has no monochromatic cycle (we can by minimality). Then G_1 must be in $\mathcal{R}(\mathcal{C})$.

Proof of theorem I

Lemmas

Lemma I.1

Every $G \in \mathscr{M}(\mathscr{C})$ is 2-connected and satisfies $\delta(G) = 3$.

Lemma I.2

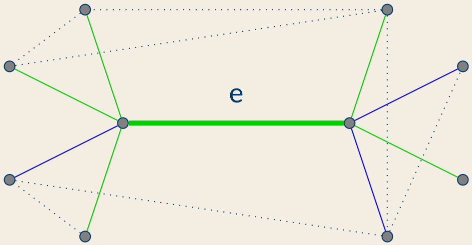
Let $G \in \mathscr{R}(\mathscr{C})$ and $e \in E(G)$ which lies in at most one triangle. Then the contracted graph G/e is in $\mathscr{R}(\mathscr{C})$.

Idea

- *Case 1.* e belongs to no triangle in G.

Proof of Lemma I.2

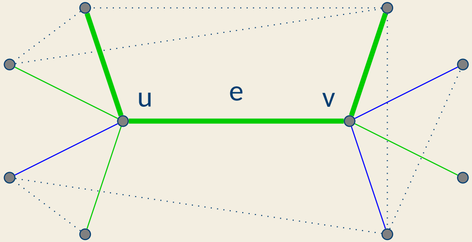
Case 1: e is in no triangle



- 2-edge colouring of G/e $\longleftrightarrow$ 2-ed. colouring of $G - e$.
- Any monochr. cycle in $G - e$ induces one monochr. cycle in G/e.

Proof of Lemma I.2

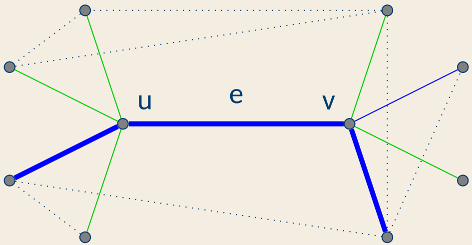
Case 1: e is in no triangle



- If no monochromatic cycle in $G - e$, any colour we paint e of would generate a cycle in G.
- $G - e$ contains a blue and a green path between u and v.

Proof of Lemma I.2

Case 1: e is in no triangle



- If no monochromatic cycle in $G - e$, any colour we paint e of would generate a cycle in G.
- $G - e$ contains a blue and a green path between u and v.

Proof of theorem I

Lemmas

Lemma I.1

Every $G \in \mathscr{M}(\mathscr{C})$ is 2-connected and satisfies $\delta(G) = 3$.

Lemma I.2

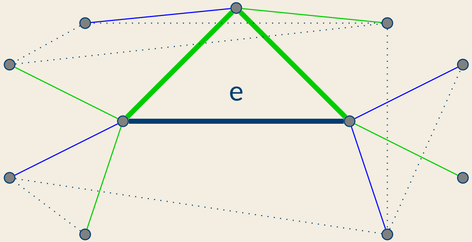
Let $G \in \mathscr{R}(\mathscr{C})$ and $e \in E(G)$ which lies in at most one triangle. Then the contracted graph G/e is in $\mathscr{R}(\mathscr{C})$.

Idea

- *Case 1.* e belongs to no triangle in G.
- *Case 2.* e belongs exactly to one triangle in G.

Proof of Lemma I.2

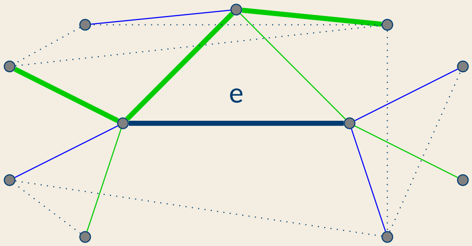
e belongs exactly to one triangle in G



- 2-edge colouring of G/e $\longleftrightarrow$ 2-ed. colouring of $G - e$ with the other two edges of the same colour.
- If no monochromatic cycle in $G - e$, procede as above.

Proof of Lemma I.2

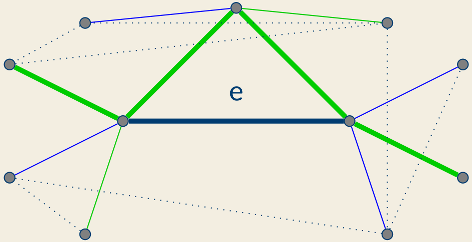
e belongs exactly to one triangle in G



- If the monochr. cycle in $G - e$ use one edge we are done.

Proof of Lemma I.2

e belongs exactly to one triangle in G



- If the monochr. cycle in $G - e$ use one edge we are done.
- If the monochr. cycle in $G - e$ use both edges then one path joins u and v.

Proof of theorem I

Theorem I

For every $G \in \mathscr{M}(\mathscr{C})$ we have:

$$\{K_5 - e, K_4 \vee K_4\} \ni G_0 \prec \cdots \prec G_n = G$$

Lemmas

- I.1 Every $G \in \mathscr{M}(\mathscr{C})$ is 2-connected and satisfies $\delta(G) = 3$.
- I.2 Let $G \in \mathscr{R}(\mathscr{C})$ and $e \in E(G)$ which lies in at most one triangle. Then the contracted graph G/e is in $\mathscr{R}(\mathscr{C})$.
- I.3 Any 2-connected graph G with every edge contained in at least two triangles satisfies $e(G) \geq 2v(G)$, unless $v(G) \leq 6$.

Another construction

Theorem II

Let $G \in \mathscr{M}(\mathscr{C})$, then also $G^* \in \mathscr{M}(\mathscr{C})$, where G^* is constructed in one of the following ways:

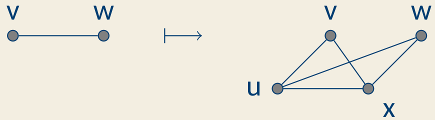
Move I

Another construction

Theorem II

Let $G \in \mathscr{M}(\mathscr{C})$, then also $G^* \in \mathscr{M}(\mathscr{C})$, where G^* is constructed in one of the following ways:

Move II

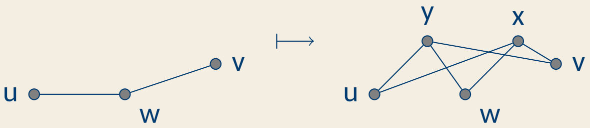


Another construction

Theorem II

Let $G \in \mathscr{M}(\mathscr{C})$, then also $G^* \in \mathscr{M}(\mathscr{C})$, where G^* is constructed in one of the following ways:

Move III



Proof of theorem II

Theorem II

Let $G \in \mathscr{M}(\mathscr{C})$, then also $G^* \in \mathscr{M}(\mathscr{C})$, where G^* as above.

Move I or II.

Idea

We simply prove the conditions stated by the arboricity theorem for G^*. I.e. it is an exercise to prove that:

$$\frac{e(G^*)-1}{v(G^*)-1} = 2 \qquad \text{and} \qquad \frac{e(H)-1}{v(H)-1} < 2$$

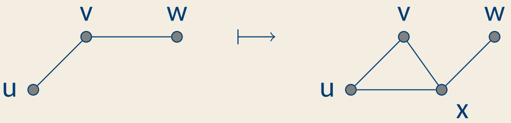
As a corollary we get that $\#\mathscr{M}(\mathscr{C}) = \infty$.

Proof of theorem II

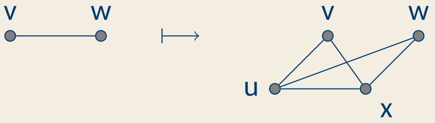
Theorem II

Let $G \in \mathscr{M}(\mathscr{C})$, then also $G^* \in \mathscr{M}(\mathscr{C})$, where G^* is as above.

✓



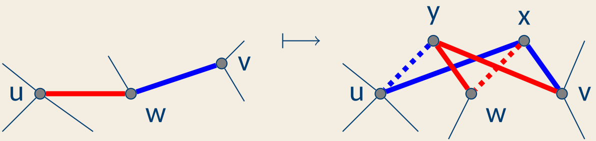
✓



Proof of theorem II

Theorem II

Let $G \in \mathscr{M}(\mathscr{C})$, then also $G^* \in \mathscr{M}(\mathscr{C})$, where G^* is as above.

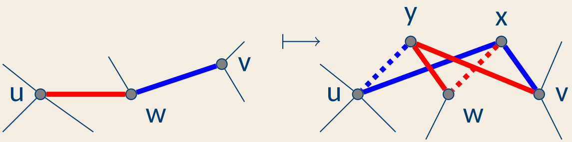


- $e(G^*) = e(G) + 4$, $v(G^*) = v(G) + 2$, therefore $G^* \in \mathscr{R}(\mathscr{C})$.
- To prove minimality, consider $G^* - e$ for a generic $e \in G^*$.

Proof of theorem II

Theorem II

Let $G \in \mathscr{M}(\mathscr{C})$, then also $G^* \in \mathscr{M}(\mathscr{C})$, where G^* is as above.



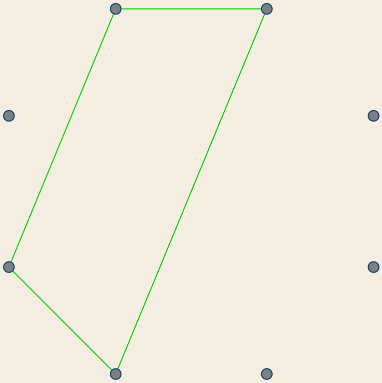
- If $e \notin E(G)$ (it has been added) argue as in the previous lemma.
- If $e \in E(G)$ set a colouring of $E(G) - e$. If this colouring has no monochromatic cycles neither has our colouring of $G^* - e$,

Forests of cycles

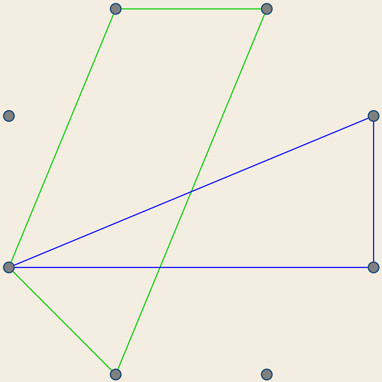
Def: forest of cycles

A forest of cycles is a graph F which can be obtained by starting with a cycle and then recursively adjoining further cycles s.t. any new cycle has at most one vertex in common with the previous graph.

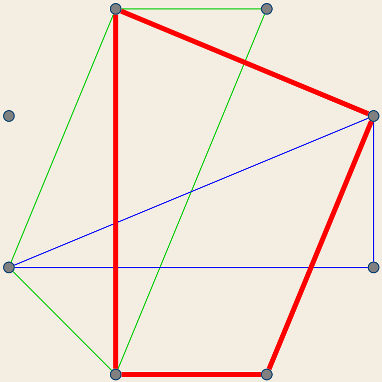
Forests of cycles



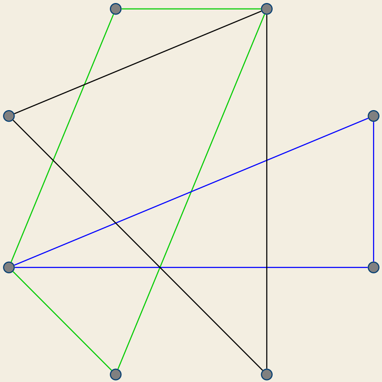
Forests of cycles



Forests of cycles



Forests of cycles



Forests of cycles

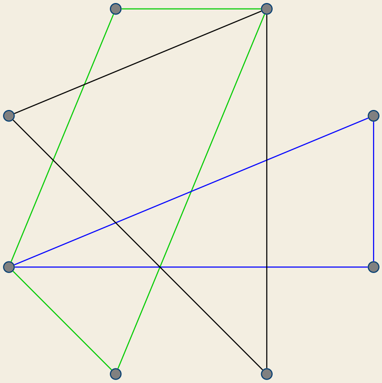
Def: forest of cycles

A forest of cycles is a graph F which can be obtained by starting with a cycle and then recursively adjoining further cycles s.t. any new cycle has at most one vertex in common with the previous graph.

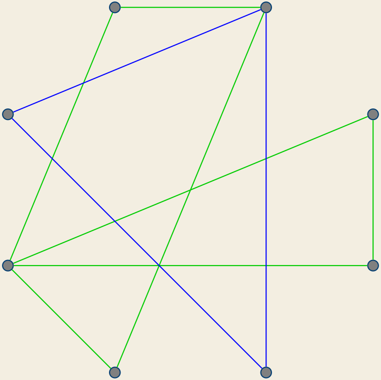
Note:

- We can construct forests of cycles with arbitrarily large maximum degree.
- Each edge of F belongs **exactly** to one cycle. So we can edge-colour F in such a way that each cycle is monochromatic, and choose the colour of each cycle freely.

Forests of cycles



Forests of cycles



Forests of cycles

Theorem III

For every forest of cycles F and every integer $n \geq 5$ satisfying $n \geq |F|$ there exists $G \in \mathscr{M}(\mathscr{C})$ with the following properties:

(1) $|G| = n$.

(2) F is a subgraph of G.

(3) Every cycle-monochromatic 2-edge-colouring of F extends to a 2-edge-colouring of G, in which there are no monochromatic cycles other than those already in F.

Concluding remarks

What we saw

Possibility to construct $\mathscr{M}(\mathscr{C})$ from some base graphs splitting some vertex. And other constructions inside $\mathscr{M}(\mathscr{C})$.

Concluding remarks

What we saw

Possibility to construct $\mathscr{M}(\mathscr{C})$ from some base graphs splitting some vertex. And other constructions inside $\mathscr{M}(\mathscr{C})$.

Questions:

- Can we find $G \in \mathscr{M}(\mathscr{C})$ with arbitrary girth?
- Can we study $\mathscr{M}(\mathscr{C}_k)$? Maybe finding some similar result?
- Edge decomposition in subgraphs with at most k cycles.
- Study the relation between $\mathscr{R}(\mathscr{C})$ and $\mathscr{R}(K_3)$.

Bibliography

[1] Damian Reding and Anusch Taraz, *Minimal Ramsey Graphs for cyclicity*, arXiv:1807.11890v1, Jul 2018.

Thank you for your attention