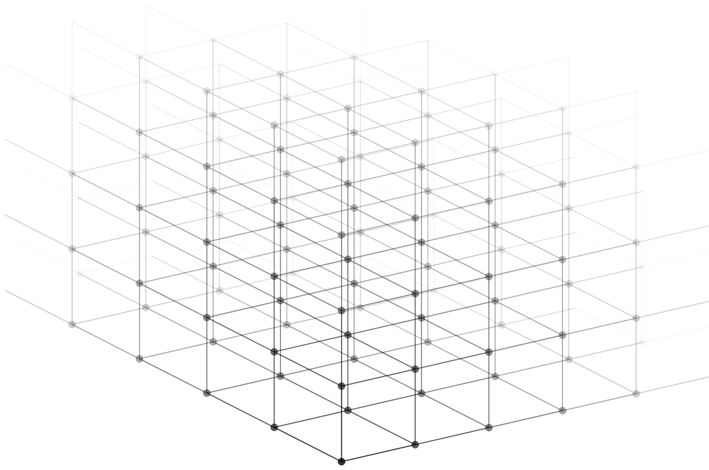
ETH zürich

SEMESTER PAPER IN COMBINATORICS

Department of Mathematics

Separator Theorems



Supervisors:
Prof. Benny Sudakov
Dr Istvan Tomon

Candidate:
Domenico Mergoni

Academic Year 2019-20

Abstract

Given a graph G, a subset S of its vertex set is called a separator if $G \setminus S$ does not have connected components of size bigger than $2n/3$. Separators are an object of great interest both from a theoretical and an algorithmic point of view. In this paper we analyze separators for selected families of graphs, inspired by the geometric model of intersection graphs of ground configurations. Along the way we give relevant examples, study the possible extensions of the various results and present some of the main tools for dealing with separators.

Contents

1 Introduction **2**
- 1.1 Separators in literature . . . 2
- 1.2 Ground configuration and graph avoidance . . . 2
- 1.3 Some extremal result from literature . . . 3
- 1.4 The ordered case and our results . . . 4

2 Some first result about separators **5**

3 Extremal Theory and Separators **8**

4 Construction of separators **9**

5 Variations to the previous methods **11**
- 5.1 The induced case . . . 11
- 5.2 Two counterexample . . . 12

6 Concluding remarks **13**

1 Introduction

A *separator* of a graph G is a subset S of its vertex set such that no connected component of $G \setminus S$ has size bigger than $2|G|/3$. Separators have been studied extensively both from a practical and a theoretical point of view [12], [2], [8].

Indeed, a useful method in solving algorithmically many kinds of graph-related problems is the so-called "divide and conquer" [1]. In this method, the problem of interest is divided into two or more smaller problems, which are then solved recursively; when the recursion stops, the smaller solutions are combined to solve the original problem. For this method to be efficient, each subproblem must be significantly smaller than the original one. One way to guarantee that this happens is to make all subproblems roughly the same size, and hence the importance of separators.

1.1 Separators in literature

One of the first results about separators is due to Lipton and Tarjan, who proved in 1977 [12] that any planar graph on n vertices has a separator of size $\sqrt{8n}$. In their article, geometric properties of planar graphs are used to explicitly construct such separator; for example, Jordan's curve theorem is used to separate the planar graph along cycles. Because geometry is used in important points of the proof of Lipton and Tarjan, it is surprising that Alon, Seymour and Thomas [2] were able to prove using only combinatorial means that a much more general theorem holds.

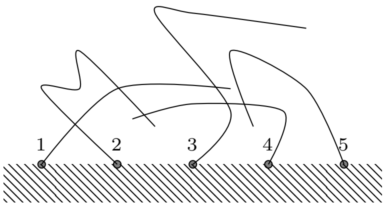
Theorem 1.1 (Alon, Seymour and Thomas, [2]). *Let H be a simple graph on h vertices, and let G be a graph on n vertices with no H-minor. Then there exists a separator of G of size at most $h^{3/2}n^{1/2}$.*

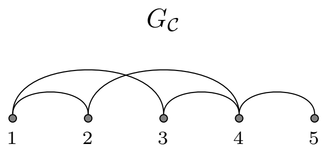
This theorem is indeed an extended version of Lipton and Tarjan theorem, because of Kuratowski's theorem; moreover, it also answers the separator problem for graphs embedded in other surfaces because of the characterization of such graphs due to Robertson and Seymour [15]. But the link established by Lipton and Tarjan between graph separators and geometry survives, as it is shown by many recent articles among which two by Pach and Fox [8], [9]; this is because many classes of graphs which have small separators arise from geometric problems or have strong geometric properties as in the theorem by Lipton and Tarjan. One class of special interest is the one of intersection graphs; given a family of geometrical objects $\mathcal{C}$, it is natural to define its *intersection graph* as the graph with vertex set $\mathcal{C}$ and with edges that connects intersecting objects; it is often the case that graphs defined in this way have separators which are, in some sense that we are going to define, small.

1.2 Ground configuration and graph avoidance

As we said, intersection graphs are an interesting family of graphs when studying separators, and surely among them *string graphs*, which are intersection graphs of a set of curves in the plane, hold a special position (by curve in a plane we mean the image of a continuous, injective map $c : [0, 1] \to \mathbb{R}^2$; we say that c starts at $c(0)$). Our topic of interest is better explained by an example.

Example 1.2 (Ground configuration). In the semiplane $\mathbb{R} \times \mathbb{R}^+$, consider a set of curves $\mathcal{C} = \{\ell_1, \ldots, \ell_n\}$ such that ℓ_i has the point of coordinates $(i, 0)$ as a starting point. Let $G_{\mathcal{C}}$ be the intersection graph of $\mathcal{C}$; as in the following figure.





Even if the complete graph on n vertices K_n is a particular case of intersection graph of ground configuration and it does not have small separators, if we consider the number of edges we are still able to prove that graphs defined in this way have separators which are in some sense small. It would be interesting to justify combinatorially the presence of small separators in this kind of graphs, and we are going to do that to some extent.

We give precise definitions in the next sections, but one remark that can help us understand how we may proceed is the following. Consider the intersection graph $G_{\mathcal{C}}$ of a ground configuration and suppose $(1, 3), (2, 4) \in G_{\mathcal{C}}$, as it is in the previous figure. Then by Jordan's curve theorem, $(\ell_2 \cup \ell_4) \cap (\ell_1 \cup \ell_3) \neq \emptyset$, and therefore there is at least another edge between the considered four vertices. Equivalently, the intersection graph of a ground configuration avoids as an induced subgraph if we consider the ordering of the vertices.

Our main focus in this paper will be studying separators of graphs avoiding specific subgraphs.

1.3 Some extremal result from literature

The following theorem by Lipton, Rose and Tarjan [11], that we are going to prove following the steps of Fox and Pach [8], tells us more about the correlation between graph separators and extremal functions.

Theorem 1.3 (Lipton, Rose, Tarjan [11]). *Let $\mathcal{G}$ be a family of graphs closed under taking induced subgraphs and let $\varepsilon > 0$. Suppose every G in $\mathcal{G}$ has a separator of size $O\left(\frac{|G|}{(\log(|G|))^{(1+\varepsilon)}}\right)$. Then there exists a constant M such that every graph in $\mathcal{G}$ on n vertices has at most Mn edges.*

Therefore, given a family of graphs $\mathcal{G}$ closed under taking subgraphs, if we already know that there are elements in $\mathcal{G}$ with more than linearly many edges, then Theorem 1.3 allows us to conclude that not all the members of $\mathcal{G}$ have small separators. This can be useful sometimes to direct our attention, but we have to be careful not to believe that the converse is also true. Indeed, in general, it is not true that families graphs with linearly many edges have separators of sublinear size, as Erdös, Graham and Szemerédi proved.

Theorem 1.4 (Erdös, Graham and Szemerédi [5]). *For every $\varepsilon > 0$ there exists $c = c(\varepsilon)$ such that almost all graphs G with $(2+\varepsilon)k$ vertices and ck edges have the property that after the omission of any k of its vertices, a connected component of at least k vertices remains.*

1.4 The ordered case and our results

Before continuing, and to avoid any possible confusion, we now introduce some terminology.

Definition 1.5. An *ordered graph* is a simple graph $G = (V, E)$ with a total order defined on its vertex set; because of the uniqueness of finite total orders, we assume $V = [n]$. Given H, H' ordered graphs on the same number of vertices, we say that H is *isomorphic* to H' if the monotone bijection between $V(H)$ and $V(H')$ is a graph isomorphism. Given H, G two ordered graphs, we call H a *subgraph* of G if there exists a monotone injective function $f : V(H) \to V(G)$ such that $(x, y) \in E(H)$ implies $(f(x), f(y)) \in E(G)$; in the particular case in which f is an isomorphism of ordered graphs between H and $G[f(V(H))]$ we say that H is an *induced subgraph* of G.

We are also going to use often the following notation.

Notation. For an ordered graph H, we denote with $\mathcal{A}_H^{<,I}$ the family of graphs which avoid H as an induced, ordered, subgraph; similar symbols such as $\mathcal{A}_H^{<}$, $\mathcal{A}_H^{I}$ and $\mathcal{A}_H$ are also used, respectively for the family of ordered graphs without H as a subgraph and for the families of unordered graphs which avoids H as an induced subgraph or simply as a subgraph.

With this notation, we can easily reformulate the remark we did about the ground configuration example; we were simply pointing out that if G is the intersection graph of a ground configuration, then $G \in \mathcal{A}_{\frown}^{<,I}$.

One natural question in this setting is whether this kind of separator result also holds for $\mathcal{A}_H^{<,I}$ or $\mathcal{A}_H^{<}$ for general ordered graphs H. The next theorem, which is a variation of a result by Erdös, Stone and Simonovits [7], [6], answers in the negative.

Before introducing the result, we define the *interval chromatic number* of an ordered graph H as the minimum number of intervals $\chi_<(H)$ in which the vertex set of H can be divided in such a way that no edge has its two vertices in the same interval. If $\chi_<(H) = 2$ we say that H is bipartite.

Theorem 1.6 (Pach and Tardos, [14])**.** *Let H be an ordered graph and $r + 1 = \chi_<(H)$ its interval chromatic number. We have:*

$$ex_<(n, H) = \left(1 - \frac{1}{r}\right)\frac{n^2}{2} + o(n^2).$$

This theorem, combined with Theorem 1.3, implies that there is no chance of finding separators of sublinear size for the family $\mathcal{A}_H^{<}$, unless H bipartite; in which case, Theorem 1.6 only tells us that $ex_<(n, H)$ is $o(|G|^2)$. Under further assumptions, we can say more about $ex_<(n, H)$.

Theorem 1.7 (Corollary of a result by Marcus and Tardos [13])**.** *Let M be an ordered bipartite matching (an ordered matching that is bipartite as an ordered graph). Then*

$$ex_<(n, M) = O(n).$$

Strong of these premises, we are going to prove the existence of separators in some particular cases. The result is as follows.

Theorem 1.8. *a) Each $G \in \mathcal{A}^{<}_{\frown\!\!\frown}$ has a separator of size 2.*

b) Each $G \in \mathcal{A}^{<}_{\overset{\frown}{\frown}}$ on n vertices has a separator of size at most $4\sqrt{n}$.

Theorem 1.8 does not prove our original claim that ground configuration graphs have small separators. In order to do so, we have to modify our statement (and most importantly, its proof) to hold for the class $\mathcal{A}^{<,I}_{\frown\!\!\frown}$. More generally, it would be interesting to understand when it is possible to extend separator results in $\mathcal{A}^{<}_{H}$ to theorems about separators of $\mathcal{A}^{<,I}_{H}$. As we mentioned before, a different notion of small separator is needed when dealing with the induced case, because the complete graph K_n only has complete graphs as induced subgraphs. Therefore, any separator result for families in the form $\mathcal{A}^{<,I}_{H}$ has also to depend on some other parameter.

Anyway, as we are going to see, the method used to find separators for $\mathcal{A}^{<}_{\frown\!\!\frown}$ can also be used to study separators of $\mathcal{A}^{<,I}_{\frown\!\!\frown}$ with some success.

Theorem 1.9. *Let $G \in \mathcal{A}^{<,I}_{\frown\!\!\frown}$ with n vertices and m edges. There exists a separator for G of size $O(\sqrt{m})$.*

Because the method used for $\mathcal{A}^{<}_{\frown\!\!\frown}$ seems quite versatile, a natural question is whether we can solve similar problems both in the induced and not induced case. We denote with R_k the "rainbow" with k edges, i.e. the ordered bipartite graph over $[2k]$ in which $i \sim n+1-i$. By Theorem 1.7 we know that, for any k, graphs in $\mathcal{A}^{<}_{R_k}$ have at most linearly many edges; still, for big k we can expect graphs in $\mathcal{A}^{<}_{R_k}$ without small separators, as the following result shows.

Theorem 1.10. *There exist graphs in $\mathcal{A}^{<}_{R_{k+1}}$ on N vertices with no separator of size $N^{1-1/k}$.*

Indeed, let $Q_k(n)$ be the k-dimensional grid of size n (over the vertex set $[n] \times \ldots \times [n]$, with n^k vertices). As we are going to show, there exists an ordering of the vertices of $Q_k(n)$ such that the resulting graph $G_k(n)$ is in $\mathcal{A}^{<}_{R_{k+1}}$. By a result of Bollobás and Leader [4], the size of the smallest separator of $Q_k(n)$ is $\Theta(n^{k-1})$.

One last example of how it is not always possible to generalize a result from $\mathcal{A}_H$ to $\mathcal{A}^{I}_{H}$ is the following.

Theorem 1.11. *There exists $G \in \mathcal{A}^{I}_{\pitchfork}$ with $n^{3/2}$ edges and without separators of sublinear size.*

Paper organization. The rest of the paper is organized as follows. In the next section, we present separator result for some examples of general interest such as trees. In section 3 we present some extremal result for certain families of ordered graphs. In section 4 we start the construction of separators for the families of graphs in which we are interested, in the unordered case. In section 5 we introduce the induced case and we present some modifications to the cases analyzed in the Section 4. In the last section, we present some open problems and some concluding remarks.

2 Some first result about separators

In this section, our goal is to familiarize with the concept of separator by showing some meaningful examples. But before we start, a clarification is needed; indeed, in the literature, often a different definition of separator is given. We want to settle any confusion with the following remark.

Remark 2.1. Let $G = (V, E)$ be a graph on n vertices, $S \subseteq V$. The two following conditions are equivalent.

- All connected components of $V \setminus S$ have size at most $2n/3$.
- $V \setminus S = A \sqcup B$ where $|A|, |B| \leq 2n/3$ and there are no edges between A and B.

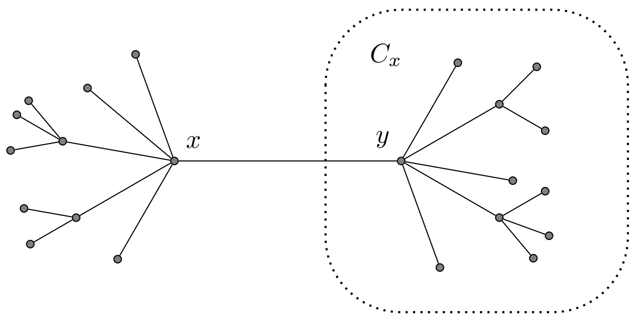
Often the second condition is used as definition of separator.

A class of graphs which behaves extremely well with respect to separators is the class of trees. This should not surprise: in a tree, every vertex is a vertex cut, and hence small separators should be expected.

Proposition 2.2. *Let T be a forest. Then T has a separator of size at most* 1.

Proof. Without loss of generality, we can assume T is a tree because if this is not the case at most one connected component of T can have size bigger than $2|T|/3$; separating this connected component would separate the whole forest.

For $x \in V(T)$ let us denote with C_x the biggest connected component of $T \setminus x$. Let x_0 be such that for any $y \in N(x_0)$ we have $|C_y| \geq |C_{x_0}|$. Assume by contradiction $|C_{x_0}| > 2|T|/3$ and let y be the only neighbour of x_0 in C_{x_0}.



There are two possible cases:

a) $C_y \subset C_{x_0}$, but in this case $|C_y| < |C_{x_0}|$, which is a contradiction.

b) $C_y \subset \{x_0\} \cup (V(T) \setminus C_{x_0})$. In this case, $|C_y| \leq |T|/3$ which is again a contradiction.

□

This is a first, easy example of a subgraph-avoiding family which has small separators. Indeed forests are graphs avoiding cycles, and we proved here that they have separators of just one vertex. There is another example in which a similar result holds.

Proposition 2.3. *Let $G \in \mathcal{A}_{\curlywedge}$. Then G has a separator of size at most* 2.

Proof. The maximal degree of G is at most 2 and hence G is vertex-disjoint union of paths and cycles. If every connected component of G has size smaller than $2|G|/3$, then the empty set is a separator. If this is not the case, let C be the connected component with size bigger than $2|G|/3$. C is either a cycle or a path; in both cases, it has a separator of size at most 2. □

Even if these results may let us think that graph avoidance is a property which allows us to have small separators quite often, this is not the case. Indeed, there are also many results in the other direction. Among all, we present two for their relevance.

But before proving the next result, we need a Lemma of an unknown author that we found on [16].

Lemma 2.4. *Let G be an α-expander on n vertices, and let S be a separator in G; then $|S| \geq \frac{\alpha n}{3(1+\alpha)}$. In particular, expander graphs do not have separators of sublinear size.*

Proof. Let S be a separator of size s separating A of size a and B of size b; we may assume $a \leq b \leq 2n/3$ and hence $a + s \geq n/3$. Because $N(A) \setminus A \subseteq S$, we must have $s - \alpha a \geq 0$ by definition of α-expander. Dividing by α and adding $a + s \geq n/3$ we obtain our result. □

Proposition 2.5. *Not every graph in $\mathcal{A}_{\pitchfork}$ have separators of sublinear size.*

Proof. Following one of the exercises of a book by Alon and Spencer [3], we are going to prove that there exists $c' > 0$ and $G \in \mathcal{A}_{\pitchfork}$ such that G is a c' expander. Let π_1, π_2, π_3 be three random permutations between two disjoint sets A and B, both of size n; consider the random bipartite graph with vertiex set $A \sqcup B$ and with edge set $\{(a, \pi_i(a)) : a \in A, i \in 1, 2, 3\}$. We want to prove that G is a c'-expander for some $c' > 0$ with positive probability; to do so, it suffices to show that, with positive probability, for some $c > 0$, any $L \subset A$ of size at most $n/2$ has $|N(L)| \geq c\,|L|$ (this is because for any set $S \subset V$ of size $\leq n$ we can assume $S \cap A \geq S \cap B$, and in the case $S \cap A \geq n/2$ we have anyway that $N(S) \setminus S \geq cn/2$).

Therefore, fix $L \subseteq A, |L| \leq n/2$. To simplify the notation we use: $\ell = |L|\,,\ k = (1+c)\ell,\ m = c\ell$. Then:

$$\mathbb{P}\left[N(L) \leq k\right] \leq \frac{\binom{m}{k}\frac{k!}{m!}^{3}}{n!^3} = \frac{n!k!^3}{k!(n-k)!m!^3n!^2} = \frac{k!^2}{(n-k)!m!^3n!^2}.$$

Because we have to consider the event for any L of size less or equal than $n/2$, we have to bound:

$$\mathbb{P}\left[\exists L : N(L) \leq k\right] = \sum_{\ell=1}^{n/2} \binom{n}{\ell} \frac{k!^2}{(n-k)!m!^3n!^2} \leq \sum_{\ell=1}^{n/2} \frac{k!^2}{\ell!(n-\ell)!(n-k)!m!^3n!}$$
$$\leq \frac{n}{2} \max_{\ell \leq \frac{n}{2}} \frac{((1+c)\ell)!^2}{\ell!(n-\ell)!(n-(1+c)\ell)!(c\ell)!^3n!}.$$

Which is strictly smaller than 1 for some $c > 0$ small enough. Therefore with positive probability our graph is expander. By construction it also is in $\mathcal{A}_{\pitchfork}$. □

The following example shows how similar families can have different separator results. We do not prove this result, which can be proved using the probabilistic method and Theorem 1.3.

Proposition 2.6. *There exist graphs without k-cycles as induced subgraphs and without small separators.*

At this point it should be clear that, given a graph H, it is not trivial to decide whether $\mathcal{A}_H$ has small separators or not.

3 Extremal Theory and Separators

We said that the study of the separators of a family of graphs is strongly related to the study of its extremal properties. In this section, we present some results of extremal theory that are relevant for us.

For any graph H, the family $\mathcal{A}_H$ is closed under taking subgraphs. This is also true if H is ordered and we consider $\mathcal{A}_H^<$. Hence Theorem 1.3 is of great interest in our case, as we already saw for example in Proposition 2.6.

Proof of Theorem 1.3. Let $G = (V, E)$ be in $\mathcal{G}$ with n vertices and average degree d, and denote $\phi(n) = \frac{1}{\log(n)^{1+\varepsilon}}$. Let n_0 such that $\phi(n_0) \leq \frac{1}{12}$ and assume $n > n_0$ (we can do it because we are only interested in the behaviour for large n, as we can set the constant M as we like). Because we assume that any graph in $\mathcal{G}$ on n vertices has a separator of size $\phi(n)n$ and because of remark 2.1, we can write $V = S \cup A \cup B$ with $|S| \leq n\phi(n)$, $|A|, |B| \leq 2n/3$ and no vertices between A and B. Let d' and d'' be the average degrees respectively of $G[S \cup A]$ and $G[S \cup B]$. Because in G there are no edges between A and B, every edge of G is contained in one of these graphs. Therefore:

$$d'(|S| + |A|) + d''(|S| + |B|) \geq 2|E| = d|V|$$

so that

$$d' \frac{|S| + |A|}{|V| + |S|} + d'' \frac{|S| + |B|}{|V| + |S|} \geq d \frac{|V|}{|V| + |S|}.$$

Since $|V| = |S| + |A| + |B|$, then $\frac{|S|+|A|}{|V|+|S|} + \frac{|S|+|B|}{|V|+|S|} = 1$ and the left hand side of the inequality is a weighted mean of d' and d''. Consequently d' or d'', and without loss of generality we assume d', is at least

$$d \frac{|V|}{|V| + |S|} \geq d \frac{1}{1 + \phi(n)}.$$

Let $G_1 = G[S \cup A]$. By assumption, $\phi(n) < 1/12$ and $|S| \leq n\phi(n)$. Therefore G_1 has at most $\phi(n)n + 2n/3 \leq 3n/4$ vertices.

We can use recursion to find a sequence of induced subgraphs $G = G_0 \supset G_1 \supset \ldots$ with the property that if G_i has n_i vertices and average degree d_i then G_{i+1} has at most $3n_i/4$ vertices and average degree at least $d_i/(1 + \phi(n_i))$. We stop with G_j if the number of vertices of G_j is less or equal than n_0. Then, the average degree of G_j is at least d/K, where K is any finite number such that:

$$\prod_{i=0}^{\infty} (1 + \phi(\lceil (4/3)^i n_0 \rceil)) \leq K$$

Suppose by sake of contradiction that $d \geq K n_0$, then we have $d_j \geq d/k \geq n_0 \geq |G_j|$, which is absurd because of graph theoretical reasons. Hence we have $d < K n_0$, and therefore the number of edges of G is at most $\frac{K n_0}{2} n$. □

Therefore, families of graphs with small separators have linearly many edges. It suffices Proposition 2.5 to convince us that it is not a sufficient condition, but we are going to see another proof.

Proof of Theorem 1.4. Consider a random graph G over the vertex set V of size $n = (2+\varepsilon)k$ and with ck edges, where $c = c(\varepsilon)$ is independent from G and to be determined. There are $\binom{n}{k}$ ways in which we can choose $S \subseteq V$ of size k; we want to prove that almost surely, for any of these choices, $V \setminus S$ has a connected component of size at least k. Choose A, B one of the 2^{n-k} possible partitions of $V \setminus S$. It suffices to show that if $|A|, |B| \geq \varepsilon k$, there is an edge between A and B. Indeed, if this holds, at most εk vertices can be disconnected from the rest.

Because $|A|, |B| \geq \varepsilon k$, the probability that there are no edges between A and B is less than $\left(1 - \frac{c}{n}\right)^{\varepsilon k^2}$. We want that this event does not occur for any of the possible choices of S, A and B. Therefore, if we choose c such that

$$\binom{n}{k} 2^{n-k} \left(1 - \frac{c}{n}\right)^{\varepsilon k^2} \leq 2^{3(n-k)} \left(1 - \frac{c}{n}\right)^{\varepsilon k^2} \xrightarrow{k\to\infty} 0$$

then we obtain that almost surely G does not have separators of sublinear size. Setting c large enough ($c > 20(\varepsilon + \varepsilon^{-1})$ should suffice), we obtain:

$$\left(1 - \frac{c}{n}\right)^{\varepsilon k^2} \xrightarrow{k\to\infty} e^{-(\varepsilon c/(2+\varepsilon))k} < e^{-3(1+\varepsilon)k}.$$

□

4 Construction of separators

In this section, we present a first result towards our goal of finding separators for the intersection graph of a ground configuration. In particular, we focus on the study of separators for the classes $\mathcal{A}^{<}_{\frown\!\frown}$ and $\mathcal{A}^{<}_{\overset{\frown}{\frown}}$.

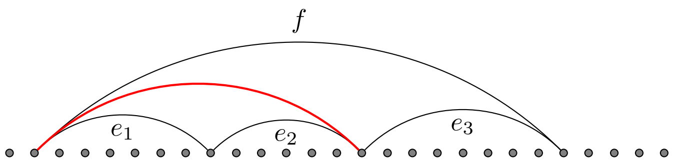
Before continuing with the proof of the first theorem, some notation is necessary.

Notation. First, recall that we consider ordered graphs to be over the vertex set $[n]$. Moreover, given two edges $e = (i_1, i_2), f = (j_1, j_2)$ in an ordered graph, we say that e is contained in f if $j_1 \leq i_1 < i_2 \leq j_2$, in this case we write $e \prec f$; the relation $\prec$ defines a partial order on the edge set. With length of an edge $e = (i, j)$ we mean $|e| = |i - j|$. Moreover, we say that $e = (i, j)$ with $i < j$ is a left edge for j and a right edge for i. Finally, we denote by $[i, j]$ the set $\{i, i+1, \ldots, j\}$.

Proof of Theorem 1.8. a) Let $G \in \mathcal{A}^{<}_{\frown\!\frown}$ on n vertices; because of the equivalence in Remark 2.1, we may assume G connected. Moreover, we can also assume that G is maximal in $\mathcal{A}^{<}_{\frown\!\frown}$ with respect to graph containment, i.e. that adding any edge to G would add a copy of $\frown\!\frown$.

Consider now an edge $(i, j) \in E(G)$ and the graph $G' = G \setminus \{i, j\}$. Because G avoids $\frown\!\frown$ as a subgraph, we have that there are no edges in G' between $[1, i-1] \sqcup [j+1, n]$ and $[i+1, j-1]$. Therefore to conclude it suffices to find an edge of length between $n/3$ and $2n/3$.

Suppose there are no such edges, i.e. that the edge set can be partitioned in the set A of those that have length strictly more than $2n/3$ ($A = \{e \in E(G) : |e| > 2n/3\}$) and the set $B = \{e \in E(G) : |e| < n/3\}$ of those that have length strictly less than $n/3$. Because by maximality of G both $(1, n)$ and $(i, i+1)$ are edges in G, this partition is non trivial. Let $f = (x, y)$ be any edge of minimal length in A and let $e_1 = (u_1^1, u_1^2), \dots, e_m = (u_m^1, u_m^2)$ be $\prec$-maximal edges in B contained in f (contained in $[x, y]$ but not contained in any other edge of B). Because G avoids , we can assume $u_1^1 < u_1^2 \le u_2^1 < u_2^2 \le \dots$. Because G_1 is maximal we have $u_1^1 < u_1^2 = u_2^1 < u_2^2 = \dots$, moreover we have $m \ge 3$ (because every edge in B is shorter than $n/2$ and $|f| > 2n/3$). But this contradicts maximality of G_1, because (u_1^1, u_2^2) is not an edge of G_1, and adding it does not add any copy of .



b) Let $G \in \mathcal{A}^{<}$ on n vertices; as in point a), we can assume G is connected and maximal. Moreover, we can notice that if $e = (x, y)$ is any edge of G, then there are no edges between $[1, x]$ and $[y, n]$. Therefore, we may assume that every edge $e = (x, y)$ with a vertex in $[n/3, 2n/3]$ is of length strictly bigger than $\sqrt{n}$. Because otherwise $S = [x, y]$ would be a separator of size $O(\sqrt{n})$. We are going to need the following claims.

Claim. 1. Because G is maximal and connected, each vertex (with the possible exception of 1 and n) has degree at least 2. In particular, each vertex has both a left and a right edge. To show that v has a left edge consider the set $D_v = \{i \in V(G) : i < v,\ \exists j > v \text{ s.t. } (i, j) \in E\}$. Then because G is connected, D_v is non empty and because G is maximal, $(v, \min D_v)$ is an edge.

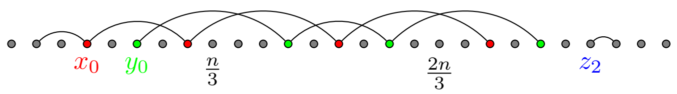
2. If $(i, j_1), (i, j_2) \in E(G_2)$ with $i < j_1 < j_2$, then by maximality of G we have that for any $j \in [j_1, j_2]$, $(i, j) \in E(G)$. The symmetric case with $j_2 < j_1 < i$ also holds.

3. By maximality of G and by point 2. of this claim, we can say that if $x_0 < y_0 < x_1 < y_1 < x_2$ (assume them sufficiently distant) are such that $(x_0, x_1), (x_1, x_2), (y_0, y_1)$ are all edges in G, then at least one of $(y_0 - 1, y_1)$ and $(y_0, y_1 - 1)$ is an edge in G. Also, at least one of $(y_0 + 1, y_1)$ and $(y_0, y_1 + 1)$ is an edge in G.

The proof proceeds as follows. Firstly, we find a relatively small set of vertices S such that $G \setminus S$ has at least two distinct connected components A and B; then we modify S to obtain S' such that A and B have roughly the same size. We are able to find easily such a construction only in the case in which every edge is sufficiently large. For this reason, we restrict our attention to $[z_1, z_2]$ for some carefully chosen z_1, z_2 which allows us to extend our result to the whole graph as follows.

Let $[z_1, z_2]$ be the maximal set containing $[n/3, 2n/3]$ such that no edge with one vertex in $[z_1, z_2]$ has length smaller than $\sqrt{n}$. For how we defined $[z_1, z_2]$ it should be clear that if $S_0 = ([z_1 - \sqrt{n}, z_1] \cup [z_2, z_2 + \sqrt{n}]) \cap [1, n]$, then in $G \setminus S_0$ the central component $[z_1, z_2]$ and the two lateral components are not connected. Indeed, there are edges strictly contained both in $[z_1 - \sqrt{n}, z_1]$ and in $[z_2, z_2 + \sqrt{n}]$, and G has no $\frown\!\!\frown$ as subgraphs. We are now ready to divide the central part $[z_1, z_2]$.

Let $x_0 = z_1$, and let x_1 be the smallest vertex such that (x_0, x_1) is an edge and, inductively, x_i the smallest vertex such that (x_{i-1}, x_i) is an edge. Let x_m be the last of those vertices in $[z_1, z_2]$ and S_{x_0} be the set $\{x_0, \ldots, x_m\}$. We have $m \leq \sqrt{n}$, because each edge is of length bigger than $\sqrt{n}$. Now consider $[z_1, z_2] \setminus S_{x_0}$; take $y_0 \in [x_0, x_1]$ and define in the same way $y_1, \ldots, y_m$ (we do all this in $[z_1, z_2] \setminus S_{x_0}$, it is possible we have to stop at y_{m-1}, but it does not change much), note that $y_i \in [x_{i-1}, x_i]$ because $G \in \mathcal{A}^{<}_{\frown}$. Let S_{y_0} be $\{y_0, \ldots, y_m\}$ and $S_1 = S_{x_0} \cup S_{y_0}$. The situation is roughly as follows.



We are now ready to partition $[z_1, z_2] \setminus S_1$. Let $A = [x_0, y_0] \cup [x_1, y_1] \cup \ldots \cup [x_m, y_m]$, $B = [y_0, x_1] \cup [y_1, x_2] \cup \ldots \cup [y_m, z_2]$. It should be clear that S_1 separates A and B in $[z_1, z_2]$.

We now have to modify S_1 to have A and B roughly of the same size. Suppose that we want to decrease the size of B (the procedure for A is similar). If $(y_0 + 1, y_1)$ is an edge in G we are done because we can take $y_0 + 1$ instead of y_0, if this is not the case, then by claim 3. we have $(y_0, y_1 + 1)$ is an edge in G. Again, if $(y_1 + 1, y_2)$ is an edge we are done, else $(y_1, y_2 + 1)$ is an edge in G. We can iterate the reasoning until we find two edges in the form $(y_i, y_{i+1} + 1), (y_{i+1} + 1, y_{i+2})$ or we substitute y_m with $y_m + 1$.

Let $S = S_0 \cup S_1$ be our separating set, then its size is smaller than $4\sqrt{n}$. □

5 Variations to the previous methods

5.1 The induced case

Finally, we can return to our original problem and study separators for the ground configuration. As we mentioned before, a quick reality check tells us we cannot expect to have a sublinear function $f(n)$ for which any element of $\mathcal{A}_H^{<,I}$ has a separator of size $f(n)$, and this is because the complete graph on n vertices K_n is in every interesting family of this kind, and it does not have separators of sublinear size. More realistically, in this section we prove the existence a separators of size $f(n, m)$ for every graph in $\mathcal{A}_H^{<,I}$ with n vertices and m edges, where $f(n, m)$ is a function sublinear in n and smaller than $\sqrt{m}$.

Proof of Theorem 1.9. As before, we can assume our graphs connected, but not maximal.

Because G has m edges, at most $\sqrt{m}$ vertices in $V(G)$ have degree bigger than $\sqrt{m}$. Therefore $S_0 = \{v \in V(G) : d(v) > \sqrt{m}\}$ has size at most $\sqrt{m}$; consider

$G' = G \setminus S_0$, take x_1 near $n/3$ in G' and let $e_1 = (w_1, w_2)$ be the longest edge which contains it. Let $S_1 = N(w_1) \cup N(w_2)$. The interesting idea here is to notice that in $G' \setminus S_1$ there are no $\frown$ in which one of the edges is e_1. We can repeat the same construction with $G' \setminus S_1$ for x_2 near $2n/3$, $e_2 = (w_3, w_4)$ the maximal edge which contains it, and $S_2 = N(w_3) \cup N(w_4)$. Notice that $S = S_0 \cup S_1 \cup S_2 \cup \{w_1, w_2, w_3, w_4\}$ has size at most $4\sqrt{m}$. Let G'' be $G \setminus S$; in G'' there are no edges between the remaining vertices of the intervals $[1, w_1], [w_1, w_2], [w_2, w_3], [w_3, w_4]$ and $[w_4, n]$ (we can suppose $w_1 < \ldots < w_4$). Therefore G'' has at least five connected components, all of size smaller than $n/3$.

□

5.2 Two counterexample

In this last part we show two families that do not have small separators.

Proof of Theorem 1.10. Let $Q_k(n)$ be the grid of dimension k on $N = n^k$ vertices. By a famous result by Bollobás and Leader [4], the smallest separator of $Q_k(n)$ has size $\Theta(n^{k-1}) = \Theta(N^{1-1/k})$. Therefore, it suffices to show that there exists an ordering $\prec$ on the vertex set of $Q_k(n)$ for which $Q_k(n)$ avoids R_{k+1}. We consider $\prec$ the alphabetic order; in particular, given two distinct $u, v \in Q_k(n)$ with $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$, we have that $u \prec v$ if the nonzero entry with the smallest index in $v - u$ is positive. What we have to show is that if $u^1 \prec \ldots \prec u^s \prec v^s \prec \ldots \prec v^1$ and $u^i \sim v^i$ in $Q_k(n)$ for any $i \in [s]$, then $s \leq k$. Let $u^i = (u^i_1, \ldots, u^i_k)$ and similarly for v^i, we define $\ell_i = \min\left\{j : u^i_j \neq v^i_j\right\}$, in particular, because $u^i \prec v^i$ we have $u^i_{\ell_i} < v^i_{\ell_i}$; moreover, because $u^i \sim v^i$ we have that

$$v^i_j = \begin{cases} u^i_j & \text{if } j \neq \ell_i \\ u^i_j + 1 & \text{if } j = \ell_i \end{cases}.$$

Therefore, if we write $\|u^i\| = \sum_{j=1}^k u^i_j$ and similarly for v^i, we have $\|u^i\| = \|v^i\| - 1$, and therefore every $\|u^i\| = \|u^h\|$ and similarly for the v^i. We conclude by proving that $i \neq j \implies \ell_i \neq \ell_j$. Suppose by sake of contradiction that $\ell_i = \ell_j = \ell$ for $i < j$; then consider that we are in one of the following cases:

a) The first non zero entry of $u^j - u^i$ has index strictly smaller than ℓ. Then because $\prec$ is the alphabetic order we would have $u^i < v^i < u^j < v^j$, which is a contradiction.

b) The first non zero entry of $u^j - u^i$ has index ℓ. Then we would have $u^i_\ell < u^j_\ell < v^j_\ell \leq v^\ell_i$ and therefore it cannot be the case $u^i_\ell = v^i_\ell - 1$.

c) The first non zero entry of $u^j - u^i$ has index h strictly bigger than ℓ. Then the ordering of the considered vectors would be $u^i \prec u^j \prec v^i \prec v^j$ because v^i and v^j would be equal on the first $h - 1$ indices, and $v^i_h < v^m_h$ because $u^i_h = v^i_h$ and $u^j_h = v^j_h$.

Therefore for every case we can find a contradiction. □

Now we are ready to prove our last result.

Proof of Theorem 1.11. For any positive integer k, let $D_k = (V, E)$ be the graph with vertex set $[k] \times [k]$ and with $(i, j) \sim (h, m)$ if and only if $i = h$ or $m = j$. Because each vertex has degree $2(k-1)$ and there are k^2 vertices, we obtain that $|E| < |V|^{3/2}$. Moreover, suppose by contradiction that D_k contains an copy of $\pitchfork$ as an induced subgraph. Let v, w_1, w_2, w_3 be the that span $\pitchfork$ with $v \sim w_i$ the only edges; then without loss of generality, we can say that v, w_1 and w_2 are on the same column; this would imply $w_1 \sim w_2$ which contradicts our assumption.

By Lemma 2.4, it suffices to show that D_k is an α-expander for some $\alpha > 0$. For any $A \subseteq V$, we denote $\partial A = N(A) \setminus A$. Fix $A \subseteq V$ a subset of the vertex set of size at most $k^2/2$ with elements from C columns and R rows. Let $B_m = \{(i, j) : i \leq m, j \leq m\}$ and $a = \lceil \sqrt{|A|} \rceil$. Firstly, we are going to show that $|\partial A| \,/\, |A| \geq |\partial B_a| \,/\, |B_a|$; indeed it holds

$$\frac{|\partial A|}{|A|} = \frac{(k-R)k + (k-C)k + RC - |A|}{|A|} \tag{1}$$

which is minimized when R and C have the same size, under the constraint that $|A|$ is constant. Therefore, our α is:

$$\alpha \geq \min_{x \in [0, 1/\sqrt{2}]} \frac{(1-x) + (1-x))}{x^2} > 0$$

□

6 Concluding remarks

The problem of estimating the size of separators has received a lot of attention, first for planar graphs [12], then for graphs avoiding minors [2] and in general for a lot of particular cases (regular graphs, graphs with bounded genus etc) [10], [8], [9]. There are still instances of this problem which remain unsolved, and some collateral problems of interest.

Problem 1. Let M be a bipartite ordered matching. Does it exist $\varepsilon > 0$ such that any G in $\mathcal{A}^{<}_{M}$ has a separator of size $O(|G|^{1-\varepsilon})$?

This problem is very general and it is near to the best we can hope to get for ordered graphs. Indeed, we already proved that if a family in the form $\mathcal{A}^{<}_{H}$ has separators of sublinear size, then H is bipartite, and it is also known that H has to be a tree if we want sublinear separators. Moreover, the maximum degree of H has to be at most three, because of the existence of graphs in $\mathcal{A}_{\pitchfork}$ without small separators.

In particular, it might be interesting to focus firstly on some selected cases.

Problem 2. Can we find separators of sublinear size for $\mathcal{A}^{<}_{\frown}$? And what about $\mathcal{A}^{<}_{\frown\frown}$? Can we generalize these construction to R_k or I_k (the version of $\frown\frown\frown$ on $2k$ vertices)?

These problems seem more feasible. As we saw, $k-1$-dimensional grids, with the right ordering, are an example of graphs in $\mathcal{A}^{<}_{R_k}$ without separators of size smaller than $O(n^{1-1/k})$ where n is the number of vertices. Because with

$k = 2$ we proved that $O(n^{1-1/k})$ vertices were sufficient, it might be the case that this holds also for general k.

One last question is the following. What conditions do we need to impose to the graphs in $\mathcal{A}^I_{\pitchfork}$ to be able to find separators of sublinear size? In particular, is it true that any G in $\mathcal{A}^I_{\pitchfork}$ on n vertices and with $o(n^{3/2})$ edges has a separator of sublinear size?

Acknowledgements. I want to thank Dr Istvan Tomon for the time dedicated and the patience showed, and Prof. Sudakov for the supervision of this project.

References

[1] A.V. Aho, J.E. Hopcroft, and J.D. Ullman. *The design and Analysis of Computer Algorithms.* Reading, MA: Addison-Wesley, 1974.

[2] Noga Alon, Paul Seymour, and Robin Thomas. "A separator theorem for graphs with an excluded minor and its applications". In: *Proceedings of the Twenty-second Annual ACM Symposium on Theory of Computing.* STOC '90. Baltimore, Maryland, USA: ACM, 1990, pp. 293–299.

[3] Noga Alon and Joel H. Spencer. *The Probabilistic Method.* 3th. Wiley Publishing, 2008.

[4] Béla Bollobás and Imre Leader. "Compressions and isoperimetric inequalities". In: *Journal of Combinatorial Theory, Series A* 56.1 (1991), pp. 47–62.

[5] P. Erdös, R.L. Graham, and E. Szemerédi. "On sparse graphs with dense long paths". In: *Computers & Mathematics with Applications* 1.3 (1975), pp. 365–369.

[6] Paul Erdös and M. Simonovits. "A limit theorem in graph theory". In: *STUDIA SCI. MATH. HUNG* 1 (1965), pp. 51–57.

[7] Paul Erdös and A. H. Stone. "On the structure of linear graphs". In: *Bull. Amer. Math. Soc.* 52.12 (Dec. 1946), pp. 1087–1091.

[8] Jacob A. Fox and János Pach. "Separator theorems and Turán-type results for planar intersection graphs". In: 2008.

[9] Jacob Fox and János Pach. "A Separator Theorem for String Graphs and its Applications". In: *Combinatorics, Probability and Computing* 19.3 (2010), pp. 371–390.

[10] John R. Gilbert, Joan P. Hutchinson, and Robert Endre Tarjan. "A separator theorem for graphs of bounded genus". In: *Journal of Algorithms* 5.3 (1984), pp. 391–407.

[11] Richard J. Lipton, Donald J. Rose, and Robert Endre Tarjan. "Generalized Nested Dissection". In: *SIAM Journal on Numerical Analysis* 16.2 (1979), pp. 346–358.

[12] Richard J. Lipton and Robert E Tarjan. *A Separator Theorem for Planar Graphs.* Tech. rep. 2. Stanford, CA, USA, 1979.

[13] Adam Marcus and Gábor Tardos. "Excluded permutation matrices and the Stanley–Wilf conjecture". In: *Journal of Combinatorial Theory, Series A* 107.1 (2004), pp. 153–160.

[14] János Pach and Gábor Tardos. "Forbidden paths and cycles in ordered graphs and matrices". In: *Israel Journal of Mathematics* 155.1 (Dec. 2006), pp. 359–380.

[15] Neil Robertson and P.D Seymour. "Graph minors. VIII. A kuratowski theorem for general surfaces". In: *Journal of Combinatorial Theory, Series B* 48.2 (1990), pp. 255–288.

[16] *Surveys in Combinatorics 2019*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2019.

[17] Gábor Tardos. "Extremal Theory of Ordered Graphs". In: *Proc. Int. Cong. of Math. (ICM 2018)*, pp. 3219–3228.