ST455: Reinforcement Learning

Lecture 3: Elementary Solution Methods
Dynamic Programming and Monte Carlo

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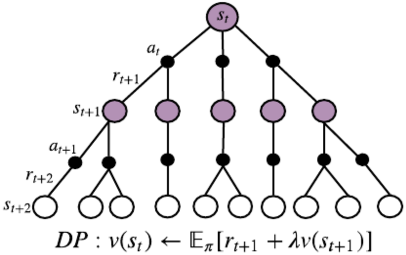
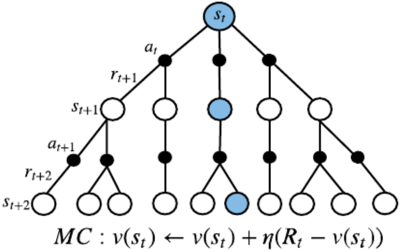
Lecture Outline

Lecture Outline (Cont'd)



Dynamic Programming (DP)



Monte Carlo (MC)

Learning v.s. Planning

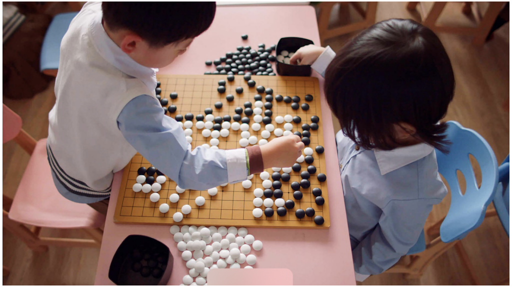
Two fundamental problems in sequential decision making

- **Planning**
  - A model of the environment (e.g., state transition, reward function) is **known**
  - The agent performs computations with its model, **without** any external interaction
  - a.k.a. deliberation, reasoning, introspection, pondering, thought, search
  - Example: **Dynamic Programming**
- **Learning**
  - The environment is initially **unknown**
  - The agent **interacts** with the model
  - The agent **learns** the optimal policy from experience
  - Example: **Monte Carlo methods**, **temporal difference learning**, **policy-based learning**, **model-based learning**

Example: Go Game

- **Planning**: Rules of Go are known
- Exhaustive search of the optimal move
- No need to play Go with others

- **Learning**: No need to know the rules
- Learn the optimal move from experience
- Practice makes perfect



Models: Finite MDPs

- Environment modelled by a finite MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$
- MDP model assumption: **Markovianity** & **time-homogeneity**
- $\mathcal{S}$: state space (a **finite** set of states)
- $\mathcal{A}$: action space (a **finite** set of actions)
- $\mathcal{P}$: state transition probability matrix, $\mathcal{P}^{a}_{ss'} = \Pr(S_{t+1} = s' | A_t = a, S_t = s)$
- $\mathcal{R}$: reward function, $\mathcal{R}^{a}_{s} = \mathbb{E}(R_t | A_t = a, S_t = s)$
- γ: discounted factor $\in [0, 1]$, allowed to be 1 if all sequences terminate (e.g., finite horizons)
- Dynamic Programming (DP) and Monte Carlo methods (MC) are **equally applicable** to settings with continuous state or action space

Bellman Equations

- Bellman equation for the (state) value function:

$$V^{\pi}(s) = \mathbb{E}^{\pi}[R_t + \gamma V^{\pi}(S_{t+1}) | S_t = s],$$

- or equivalently,

$$V^{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \Big[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^{\pi}(s') \Big].$$

- Bellman optimality equation for the **optimal** value function:

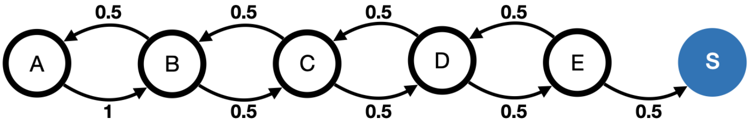
$$V^{\pi^{\text{opt}}}(s) = \max_{a} \mathbb{E}[R_t + \gamma V^{\pi^{\text{opt}}}(S_{t+1}) | A_t = a, S_t = s],$$

- or equivalently,

$$V^{\pi^{\text{opt}}}(s) = \max_{a \in \mathcal{A}} \Big[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^{\pi^{\text{opt}}}(s') \Big].$$

Bellman Equation: The Random Walk Example

- Consider a simple **random walk** on a path:

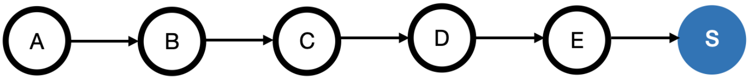


- Reward for transition to State $\boldsymbol{S}$ of value **1**, zero reward for other transitions
- Bellman equations:

$$
\begin{aligned}
V^\pi(A) &= \mathbb{E}^\pi[R_t + \gamma V^\pi(S_{t+1}) | S_t = A] = \gamma V^\pi(B) \\
V^\pi(B) &= \mathbb{E}^\pi[R_t + \gamma V^\pi(S_{t+1}) | S_t = B] = \frac{\gamma}{2} V^\pi(C) + \frac{\gamma}{2} V^\pi(A) \\
&\vdots \\
V^\pi(S) &= \mathbb{E}^\pi[R_t + \gamma V^\pi(S_{t+1}) | S_t = S] = 1
\end{aligned}
$$

Bellman Optimality Equation: Random Walk

- The **random walk** example:



- Reward for transition to State $\boldsymbol{S}$ of value $\mathbf{1}$, zero reward for other transitions
- Bellman optimality equations:

$$
\begin{aligned}
V^{\pi^{\text{opt}}}(A) &= \max_{a} \mathbb{E}[R_t + \gamma V^{\pi^{\text{opt}}}(S_{t+1}) | A_t = a, S_t = A] = \gamma V^{\pi^{\text{opt}}}(B) \\
V^{\pi^{\text{opt}}}(B) &= \max_{a} \mathbb{E}[R_t + \gamma V^{\pi^{\text{opt}}}(S_{t+1}) | A_t = a, S_t = B] = \gamma V^{\pi^{\text{opt}}}(C) \\
&\vdots \\
V^{\pi^{\text{opt}}}(S) &= \max_{a} \mathbb{E}[R_t + \gamma V^{\pi^{\text{opt}}}(S_{t+1}) | A_t = a, S_t = S] = 1
\end{aligned}
$$

State-Action Value Function

Definition

The state-action value function (better known as the **Q-function**) is expected return starting from s and a under π,

$$Q^{\pi}(s,a) = \mathbb{E}^{\pi}(G_t | A_t = a, S_t = s) = \mathbb{E}^{\pi}\left(\sum_{i=0}^{+\infty} \gamma^i R_{i+t} | A_t = a, S_t = s\right).$$

- Q^{π} is **independent** of the time t in its definition, under **time-homogeneity**
- Q^{π} is the state value V^{π} under a Markov policy that implements a at the first time and follows π afterwards
- Reduces to action value function $\mathbb{E}^{\pi}(R_t | A_t = a)$ in Lecture 1 when $\gamma = 0$, $\mathcal{S} = \emptyset$

State-Action Value Function (Cont'd)

Relationships between V^π and Q^π

- $Q^\pi \to V^\pi$:

$$V^\pi(s) = \mathbb{E}^\pi(G_t | S_t = s) = \sum_{a \in \mathcal{A}} \pi(a|s) \mathbb{E}^\pi(G_t | A_t = a, S_t = s) = \sum_{a \in \mathcal{A}} \pi(a|s) Q^\pi(s, a)$$

- $V^\pi \to Q^\pi$:

$$\begin{aligned} Q^\pi(s, a) &= \mathbb{E}(R_t | A_t = a, S_t = s) + \gamma \mathbb{E}(G_{t+1} | A_t = a, S_t = s) \\ &= \mathbb{E}(R_t | A_t = a, S_t = s) + \gamma \mathbb{E}[\mathbb{E}^\pi(G_{t+1} | S_{t+1}) | A_t = a, S_t = s] \\ &= \mathbb{E}[R_t + \gamma V^\pi(S_{t+1}) | A_t = a, S_t = s] \end{aligned}$$

Dynamic Programming

Definition (Dynamic Programming)

A collection of algorithms used to compute optimal policies given **perfect** knowledge of the environment

- **Dynamic**: sequential or temporal component to the problem
- **Programming**: optimise a "program", i.e., a policy
- Dynamic programming (DP) is **rarely** used in practice (the environment is usually unknown)
- However, they provide a foundation for other solution methods

Dynamic Programming (Cont'd)

"Dynamic programming" is used to solve many other statistical learning problems

- Learning optimal **dynamic treatment regimes** (DTRs)
- Multi-scale **change point detection**
- De Boor algorithm for evaluating **B-spline** basis functions

Also used in bioinformatics, optimisation, control theory (see wiki page)

Dynamic Programming Methods

- **Policy Iteration**: an iterative method that alternates between
  - Policy Evaluation
  - Policy Improvement

$$\pi_0 \longrightarrow V^{\pi_0} \longrightarrow \pi_1 \longrightarrow V^{\pi_1} \longrightarrow \cdots \longrightarrow \pi^{opt} \longrightarrow V^{\pi^{opt}}$$

- **Value Iteration**: simultaneously combine **policy evaluation** and **policy improvement**

$$V^{\pi_0} \longrightarrow V^{\pi_1} \longrightarrow V^{\pi_2} \longrightarrow \cdots \longrightarrow V^{\pi^{opt}} \longrightarrow \pi^{opt}$$

Policy Iteration: Policy Evaluation

- Computation of the (state) value function V^π for a given π
- According to the Bellman equation, for any s,

$$V^\pi(s) = \sum_{a\in\mathcal{A}} \pi(a|s)\left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^\pi(s')\right],$$

- written in matrix form, $V^\pi = \mathcal{R} + \gamma \mathcal{P} V^\pi$
- V^π is a column vector with one entry per state

$$\begin{bmatrix} V^\pi(1) \\ \vdots \\ V^\pi(n) \end{bmatrix} = \sum_{a\in\mathcal{A}} \begin{bmatrix} \pi(a|1)\mathcal{R}_1^a \\ \vdots \\ \pi(a|n)\mathcal{R}_n^a \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \vdots \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} V^\pi(1) \\ \vdots \\ V^\pi(n) \end{bmatrix},$$

where $\mathcal{P}_{ij} = \sum_{a\in\mathcal{A}} \pi(a|i)\mathcal{P}_{ij}^a$

Policy Evaluation (Cont'd)

- $\boldsymbol{V}^{\pi}$ is a solution of a system of $\boldsymbol{n}$ linear equations with $\boldsymbol{n}$ unknowns
- It can be computed directly

$$
\begin{aligned}
\boldsymbol{V}^{\pi} &= \mathcal{R} + \gamma \mathcal{P} \boldsymbol{V}^{\pi} \\
(\boldsymbol{I} - \gamma \mathcal{P}) \boldsymbol{V}^{\pi} &= \mathcal{R} \\
\boldsymbol{V}^{\pi} &= (\boldsymbol{I} - \gamma \mathcal{P})^{-1} \mathcal{R}
\end{aligned}
$$

- $\boldsymbol{I} - \gamma \mathcal{P}$ is **invertible** when γ is strictly smaller than $\mathbf{1}$, since

$$
\boldsymbol{x}^{\top} (\boldsymbol{I} - \gamma \mathcal{P}) \boldsymbol{x} = (\mathbf{1} - \gamma) \|\boldsymbol{x}\|_2^2 + \gamma \sum_{\boldsymbol{i},\boldsymbol{j}} \mathcal{P}_{\boldsymbol{ij}} (\boldsymbol{x_i} - \boldsymbol{x_j})^2 > \mathbf{0},
$$

when $\boldsymbol{x} \neq \mathbf{0}$. The equality holds due to that each row of $\mathcal{P}$ sums up to $\mathbf{1}$.

Policy Evaluation: Algorithm

- **Iterative Policy Evaluation**: an iterative method that outputs a sequence of value functions $V_0, V_1, V_2, \cdots, V_k \to V^\pi$
- **Initial** value function V_0 is chosen arbitrarily subject to the **constraint** that at terminal state it has value $\mathbf{0}$
- **Iterative** update rule (according to the Bellman equation):

$$V_{k+1} = \mathcal{R} + \gamma \mathcal{P} V_k$$

- **Convergence** is guaranteed when γ is strictly smaller than $\mathbf{1}$ (more in appendix), or eventual termination is guaranteed from all states under π

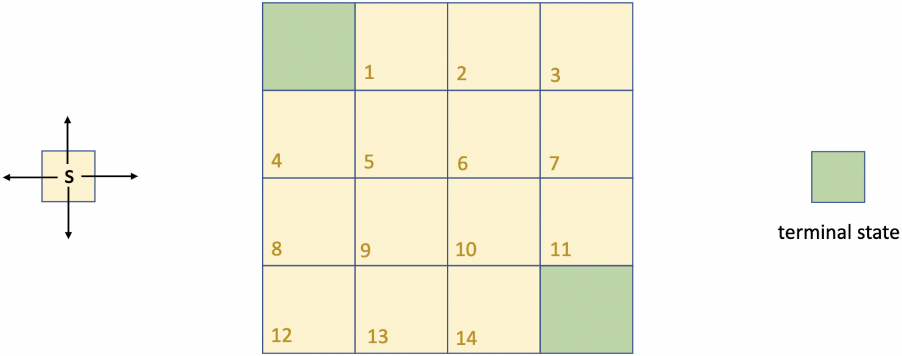
Policy Evaluation: Pseudocode

- **Input**: a policy π, a threshold parameter $\epsilon > 0$
- **Initialization**: $V(s) = 0$ for any $s \in \mathcal{S}$
- **Repeat**:

$$
\begin{aligned}
&\Delta \leftarrow 0 \\
&\textbf{For each } s \in \mathcal{S} \\
&\qquad \nu \leftarrow V(s) \\
&\qquad V(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V(s') \right] \\
&\qquad \Delta \leftarrow \max(\Delta, |\nu - V(s)|) \\
&\textbf{until } \Delta < \epsilon
\end{aligned}
$$

- **Output** V

GridWorld Example



- Undiscounted, episodic, finite MDP task
- $\mathcal{A} = \{\textbf{up, down, right, left}\}$. Actions leading out of the grid leave state unchanged
- Rewards: for each transition, the reward of value $-\mathbf{1}$

GridWorld Example (Cont'd)

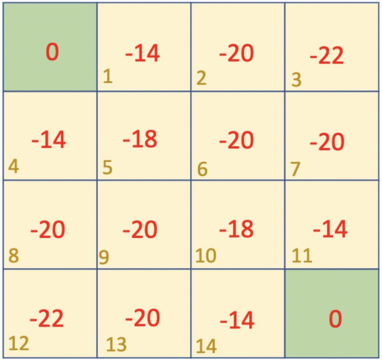


Figure: Values of uniform random policy

$$\pi(\boldsymbol{n}|\cdot) = \pi(\boldsymbol{s}|\cdot) = \pi(\boldsymbol{w}|\cdot) = \pi(\boldsymbol{e}|\cdot) = 0.25$$

GridWorld Example (Cont'd)

By **symmetry** and Bellman equation,

0	v_1 (1)	v_2 (2)	v_3 (3)
v_1 (4)	v_5 (5)	v_6 (6)	v_2 (7)
v_2 (8)	v_6 (9)	v_5 (10)	v_1 (11)
v_3 (12)	v_2 (13)	v_1 (14)	0

$$v_1 = \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_5) + \frac{1}{4}(-1 + 0) + \frac{1}{4}(-1 + v_1)$$

$$v_2 = \frac{1}{4}(-1 + v_3) + \frac{1}{4}(-1 + v_6) + \frac{1}{4}(-1 + v_1) + \frac{1}{4}(-1 + v_2)$$

$$v_3 = \frac{1}{4}(-1 + v_3) + \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_3)$$

$$v_5 = \frac{1}{4}(-1 + v_6) + \frac{1}{4}(-1 + v_6) + \frac{1}{4}(-1 + v_1) + \frac{1}{4}(-1 + v_1)$$

$$v_6 = \frac{1}{4}(-1 + v_2) + \frac{1}{4}(-1 + v_5) + \frac{1}{4}(-1 + v_5) + \frac{1}{4}(-1 + v_2)$$

$$\Rightarrow (v_1, v_2, v_3, v_5, v_6) = (-14, -20, -22, -18, -20)$$

GridWorld Example (Cont'd)

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Figure: Value functions at each iteration

Policy Iteration: Policy Improvement

- Identify some π' that is no worse than π based on V^π
- For any s, consider a hybrid policy
  - implements a at the first time
  - follows π afterwards
- Its value is given by $Q^\pi(s, a)$ (can be computed based on V^π)
- Select π' among the class of hybrid policies that **maximizes** the value

$$\pi'(s) = \arg\max_a Q^\pi(s, a)$$

- Its value is given by $Q^\pi(s, \pi'(s)) \geq V^\pi(s)$, since the hybrid policy class contains π
- Surprisingly, according to **policy improvement theorem**, $V^{\pi'}(s) \geq V^\pi(s)$ for any s!

Policy Improvement (Cont'd)

Given a policy π, improve π by acting **greedily** with respect to V^π,

$$\begin{aligned}\pi'(s) = \arg\max_a Q^\pi(s,a) &= \arg\max_a \mathbb{E}[R_t + \gamma V^\pi(S_{t+1}) | A_t = a, S_t = s] \\ &= \arg\max_a [\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^\pi(s')]\end{aligned}$$

Theorem

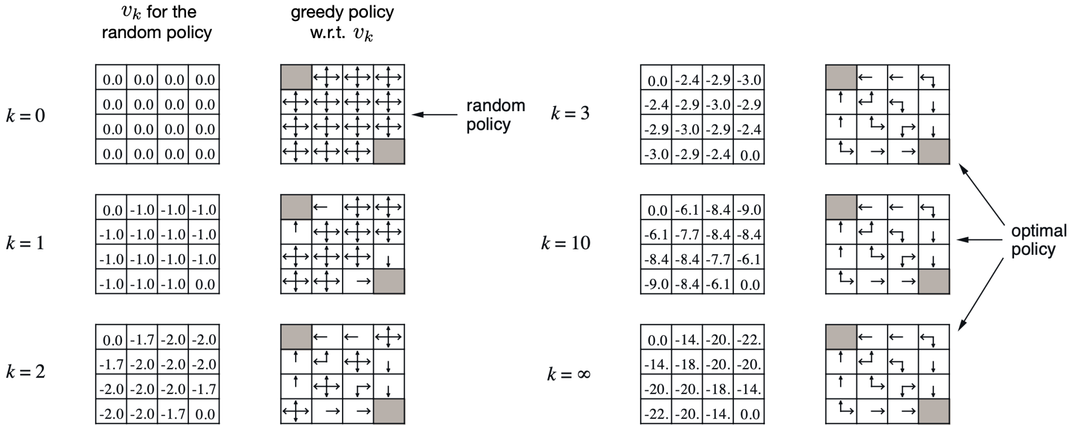
The greedy policy π' with respect to V^π is as good as or better than π,

$$V^{\pi'}(s) \geq V^\pi(s),$$

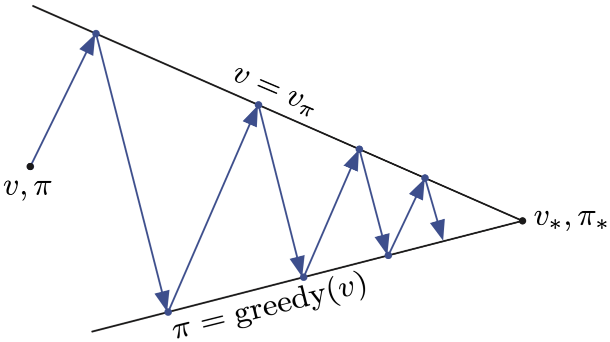
for any $s \in \mathcal{S}$.

Proof can be found in the Appendix.

GridWorld Example



Policy Iteration: Revisit



- **Policy Evaluation**: Compute $\boldsymbol{V^\pi}$ via iterative policy evaluation
- **Policy Improvement**: Generate $\boldsymbol{\pi}'$ via greedy policy improvement

Policy Iteration: Pseudocode

- **Initialization**: $V(s) = 0$, $\pi(s) \in \mathcal{A}$ arbitrarily for any $s \in \mathcal{S}$
- **Repeat**:

  $\Delta \leftarrow 0$

  For each $s \in \mathcal{S}$

  $\quad \nu \leftarrow V(s)$

  $\quad V(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V(s') \right]$

  $\quad \Delta \leftarrow \max(\Delta, |\nu - V(s)|)$

  until $\Delta < \epsilon$
- **policystable** $\leftarrow$ **True**
- **For each** $s \in \mathcal{S}$:

  $b \leftarrow \pi(s)$

  $\pi(s) \leftarrow \arg\max_a [\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V(s')]$

  If $b \neq \pi(s)$ **then policystable** $\leftarrow$ **False**
- **If policystable**, **then Return** π, **else** go to bullet point #2

Value Iteration

- Policy iteration is **computationally inefficient**, as each iteration requires executing policy evaluation which requires multiple iterations
- According to the Bellman optimality equation,

$$V^{\pi^{\text{opt}}}(s) = \max_{a \in \mathcal{A}} \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^{\pi^{\text{opt}}}(s') \right].$$

- **Value iteration** idea: iteratively apply the above updates

$$V_{k+1}(s) = \max_{a \in \mathcal{A}} \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V_k(s') \right].$$

- Drive the optimal deterministic policy

$$\pi^{opt}(s) = \arg\max_{a \in \mathcal{A}} \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^{\pi^{\text{opt}}}(s') \right].$$

- **Convergence** is guaranteed when γ is strictly smaller than $\mathbf{1}$ (more in Appendix), or eventual termination is guaranteed from all states.

Value Iteration: Pseudocode

- **Initialization**: $V(s) = \mathbf{0}$, $\pi(s) \in \mathcal{A}$ arbitrarily for any $s \in \mathcal{S}$
- **Repeat**:

  $\Delta \leftarrow \mathbf{0}$
  For each $s \in \mathcal{S}$
  $\nu \leftarrow V(s)$
  $V(s) \leftarrow \max_{a\in\mathcal{A}} \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V(s')\right]$
  $\Delta \leftarrow \mathbf{max}(\Delta, |\nu - V(s)|)$
  until $\Delta < \epsilon$
- **Output**: optimal deterministic policy given by

$$\pi^{opt}(s) = \arg\max_{a\in\mathcal{A}} \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^{\pi^{\text{opt}}}(s')\right].$$

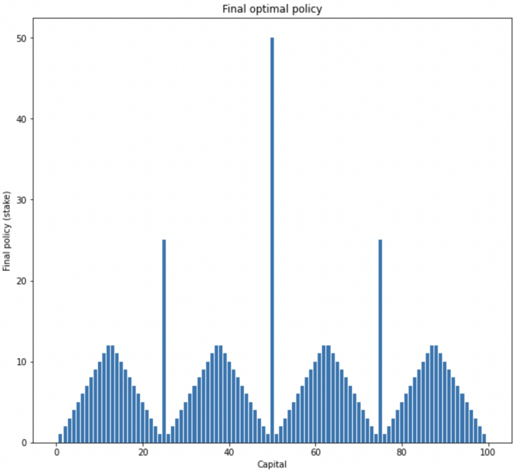
Example: Gambler's Problem

- A gambler makes bets on the outcomes of a sequence of coin flips
- The gambler must decide for each coin flip what proportion of capital to stake
- If **the outcome of the coin flip = heads**, then:
  The gambler **wins** as much money as they have staked on this flip
- Else:
  The gambler **loses** their stake
- The game ends when the gambler reaches the goal of **£100** or **runs out of money**

Example: Gambler's Problem (Cont'd)

- Undiscounted, episodic, finite MDP task
- $\mathcal{S}$: $\{0, 1, \cdots, 99, 100\}$, termination states $\mathbf{0}$ and $\mathbf{100}$
- $\mathcal{A}(s)$: $\{1, 2, \cdots, \min(s, 100 - s)\}$, depends on the state
- Pr(outcome of coin flip is heads) $= p$ (known parameter)
- Seminars:
  - Show the value function for different iterations
  - Show the optimal policy

Example: Gambler's Problem, the Optimal Policy



Some Technical Questions

- How do we know that value iteration converges to $\boldsymbol{V}^{\pi^{\text{opt}}}$?
- Or that iterative policy evaluation converges to $\boldsymbol{V}^{\pi}$?
- And therefore that policy iteration converges to $\boldsymbol{V}^{\pi^{\text{opt}}}$?
- Is the solution unique?
- These questions are resolved by **Banach fixed-point theorem** (or **contraction mapping theorem**), mentioned in Seminar 2 (more in the appendix)

Monte Carlo (MC) Methods

- Learning methods for solving the RL problem based on **averaging sample returns**
  - Estimating value functions and discovering optimal policies
  - Not assuming a model of the environment, based only on **experiences** (**model free**)
- Defined for **episodic** tasks
  - Value functions and policies are updated upon completion of an episode
  - Different from **step-by-step** methods (e.g., temporal difference learning)

MC Policy Evaluation

- **Objective**: estimate the value function V^π for a given policy π, from a set of episodes obtained by following π

$$S_0, A_0, R_0, \cdots, S_T \sim \pi$$

- V^π is the expected return $\mathbb{E}^\pi(\sum_{0 \leq t \leq T} \gamma^t R_t | S_0 = s)$
- Monte Carlo idea: use **empirical mean** return to approximate **expected return**
- Convergence is guaranteed by **law of large numbers**
- Types of MC methods:
  - **First-visit MC method**: $V^\pi(s)$ estimated by the average of returns following **each first visit** to s in a set of episodes
  - **Every-visit MC method**: $V^\pi(s)$ estimated by the average of returns following **each visit** to s in a set of episodes

First-Visit MC Policy Evaluation: Pseudocode

- **Initialization**:

  N (counter), $N(s) \leftarrow \mathbf{0}$ for all $s \in \mathcal{S}$

  Returns$(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

- **Repeat**:

  Generate an episode following policy π

  For each distinct s appearing in the episode

  $G \leftarrow$ return following the first occurrence of s

  $N(s) \leftarrow N(s) + 1$

  Returns$(s) \leftarrow$ **Returns**$(s) + G$

- **Output**:

  For each distinct s

  $N^{-1}(s)$**Returns**(s)

Every-Visit MC Policy Evaluation: Pseudocode

- **Initialization**:

  $N \leftarrow$ counter, $N(s) \leftarrow \mathbf{0}$ for all $s \in \mathcal{S}$

  Returns$(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

- **Repeat**:

  Generate an episode following policy π

  For each s appearing in the episode

  $G \leftarrow$ return following the occurrence of s

  $N(s) \leftarrow N(s) + 1$

  Returns$(s) \leftarrow$ **Returns**$(s) + G$

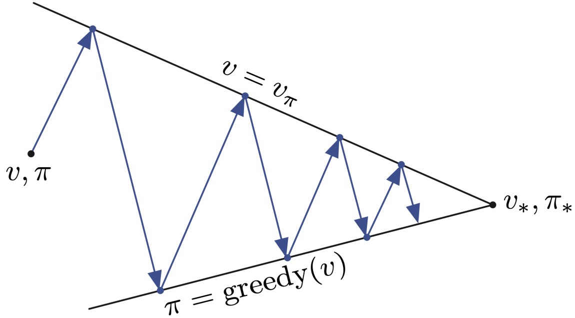
- **Output**:

  For each distinct s

  $N^{-1}(s)$**Returns**(s)

MC Control

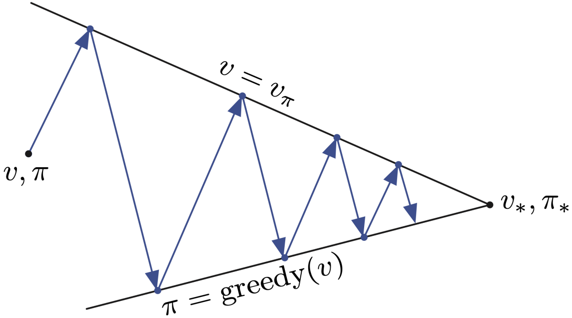
- **Objective**: use MC estimation to learn the optimal policy.
- Recall the policy iteration algorithm



- **Policy Evaluation**: Compute $\boldsymbol{V^\pi}$ via iterative policy evaluation
- **Policy Improvement**: Generate $\boldsymbol{\pi'}$ via greedy policy improvement

MC Control with Generalized Policy Iteration

- **Objective**: use MC estimation to learn the optimal policy.
- Integrate policy iteration with MC methods



- **Policy Evaluation**: Compute $\boldsymbol{V}^\pi$ via MC policy evaluation
- **Policy Improvement**: Generate π' via greedy policy improvement?

Policy Iteration Using State-Action Value Function

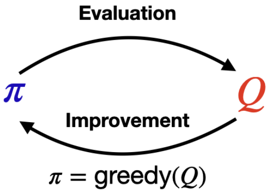
- Greedy policy improvement over V^π requires model of MDP

$$\pi'(s) = \arg\max_a [\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a V^\pi(s')]$$

- Greedy policy improvement over $Q^\pi(s, a)$ is model free

$$\pi'(s) = \arg\max_a Q^\pi(s, a)$$

MC Version of Policy Iteration



$$\pi_0 \rightarrow Q^{\pi_0} \rightarrow \pi_1 \rightarrow Q^{\pi_1} \rightarrow \cdots \rightarrow \pi^{opt} \rightarrow Q^{\pi^{opt}}$$

- **Policy Evaluation**: MC estimation of state-action value function
- **Policy Improvement**: Improve the policy wrt the current state-action value function

MC Estimation of State-Action Values

- Many state-action pairs may never be visited under a policy
  - Ex. if π is deterministic, only **one** state-action pair is observed for each distinct state
  - Need to ensure **exploration**!
- Two approaches for ensuring exploration:
  - **Exploring starts**: the first step of each episode starts at a state-action pair and every such pair has non-zero probability of being selected at the start
  - **Stochastic policies**: use policies that ensures a non-zero probability of selecting each action from the set of available actions in each given state

MC Control with Exploring Starts

- **Initialization**:

  N (counter), $N(s, a) \leftarrow 0$ for all $s \in \mathcal{S}$, $a \in \mathcal{A}$
  Returns$(s, a) \leftarrow$ an empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}$
  $\pi \leftarrow$ arbitrary
  $Q \leftarrow$ arbitrary

- **Repeat**:

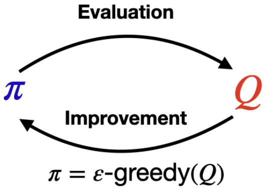
  Generate an episode using exploring starts and policy π
  For each distinct (s, a) appearing in the episode
  $\quad G \leftarrow$ return following the first occurrence of (s, a)
  $\quad N(s, a) \leftarrow N(s, a) + 1$
  $\quad \textbf{Returns}(s, a) \leftarrow \textbf{Returns}(s, a) + G$
  $\quad Q(s, a) \leftarrow \textbf{Returns}(s, a) / N(s, a)$
  $\pi(s) \leftarrow \arg\max_a Q(s, a)$ for all s

MC Control with ε-Greedy Exploration

- Simplest idea for ensuring continual exploration
- All m actions are tried with non-zero probabilities
- With probability $1 - \varepsilon$ choose the **greedy** action
- With probability ε choose an action at **random**

$$\pi(a|s) = \begin{cases} \varepsilon/m + 1 - \varepsilon, & \text{if } a = \arg\max_{a'} Q(s, a') \\ \varepsilon/m, & \text{otherwise} \end{cases}$$

MC Control with ε-Greedy Exploration (Cont'd)



$$\pi_0 \rightarrow Q^{\pi_0} \rightarrow \pi_1 \rightarrow Q^{\pi_1} \rightarrow \cdots \rightarrow \pi^{\varepsilon,opt} \rightarrow Q^{\pi^{\varepsilon,opt}}$$

Pseudocode

- **Initialization**:

  N (counter), $N(s, a) \leftarrow 0$ for all $s \in \mathcal{S}$, $a \in \mathcal{A}$

  Returns$(s, a) \leftarrow$ empty lists, for all $s \in \mathcal{S}$, $a \in \mathcal{A}$

  $\pi \leftarrow$ arbitrary ε-greedy policy

  $Q \leftarrow$ arbitrary

- **Repeat**:

  Generate an episode using exploring starts and policy π

  For each distinct (s, a) appearing in the episode

  $G \leftarrow$ return following the first occurrence of (s, a)

  $N(s, a) \leftarrow N(s, a) + 1$

  Returns$(s, a) \leftarrow$ **Returns**$(s, a) + G$

  $Q(s, a) \leftarrow$ **Returns**$(s, a) / N(s, a)$

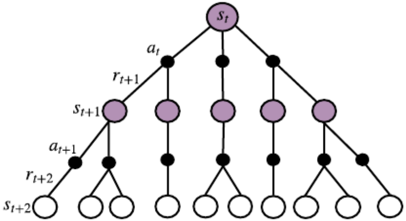
  For each distinct s:

$$\pi(a|s) \leftarrow \begin{cases} \varepsilon/m + 1 - \varepsilon, & \text{if } a = \arg\max Q(s, a) \\ \varepsilon/m, & \text{otherwise} \end{cases}$$

Summary

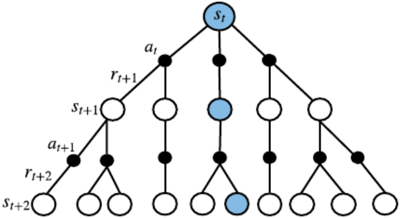
- Planning v.s. Learning
- Dynamic programming v.s. Monte Carlo Methods
- Policy Iteration v.s. Value Iteration
- Policy Evaluation v.s. Policy Improvement
- MC Policy Evaluation v.s. MC Control
- γ-Contraction, Banach Fixed Point Theorem

Summary (Cont'd)



$$DP : v(s_t) \leftarrow \mathbb{E}_\pi[r_{t+1} + \lambda v(s_{t+1})]$$

Dynamic Programming (DP)

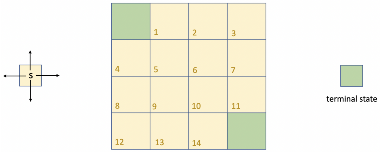


$$MC : v(s_t) \leftarrow v(s_t) + \eta(R_t - v(s_t))$$

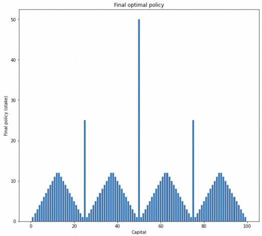
Monte Carlo (MC)

Seminar

- Solution to HW2 (due Wed 12pm)
- Iterative policy evaluation: Gridworld problem



- Value iteration: Gambler's problem



- Monte Carlo prediction & control: Black jack example

Questions

Appendix: Proof of Policy Improvement Theorem

Consider a sequence of policies:

- π_0: a given stationary policy π
- π_k: a Markov policy that implements π' at the first k times and follows π afterwards
- π_∞: the greedy policy π'

We show in the appendix

- **Step 1**: π_1 is no worse than π_0, i.e., $Q^\pi(s, \pi'(s)) \geq V^\pi(s)$
- **Step 2**: π_{k+1} is no worse than π_k for any $k \geq 1$

This proves the policy improvement theorem

Appendix: Policy Improvement Theorem, Step 1

- π_0: a given stationary policy π
- π_1: a Markov policy that implements π' at the initial time and follows π afterwards
- By definition,

$$\pi'(s) = \arg\max_{a} Q^{\pi}(s, a)$$

- This yields

$$Q^{\pi}(s, \pi'(s)) = \max_{a} Q^{\pi}(s, a) \geq \sum_{a} \pi(a|s) Q^{\pi}(s, a) = V^{\pi}(s)$$

- i.e., π_1 is no worse than π_0

Appendix: Policy Improvement Theorem, Step 2

- π_k: a Markov policy that implements π' at the first k times and follows π afterwards
- The difference between two value functions is given by

$$V^{\pi_{k+1}}(s) - V^{\pi_k}(s) = \gamma^k \mathbb{E}^{\pi'}[Q^\pi(S_k, \pi'(S_k))|S_0 = s] - \gamma^k \mathbb{E}^{\pi'}[V^\pi(S_k)|S_0 = s]$$

- Results in Step 1 yield $Q^\pi(S_k, \pi'(S_k)) \geq V^\pi(S_k)$, and hence $V^{\pi_{k+1}}(s) \geq V^{\pi_k}(s)$
- i.e., π_{k+1} is no worse than π_k

Appendix: Value Function ∞-Norm

- Measure distance between two value functions $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ by the ∞-norm
- i.e., the **largest** difference between state values,

$$\|\boldsymbol{V}_1 - \boldsymbol{V}_2\|_\infty = \max_{s \in \mathcal{S}} |\boldsymbol{V_1}(s) - \boldsymbol{V_2}(s)|$$

- Given a sequence of values $\{\boldsymbol{V_k}\}_{\boldsymbol{k}}$, convergences requires $\|\boldsymbol{V_k} - \boldsymbol{V}^*\|_\infty \to \mathbf{0}$ for some $\boldsymbol{V}^*$ as $\boldsymbol{k} \to \infty$

Appendix: Bellman Expectation Operator

Definition

Define the Bellman Expectation Operator $\boldsymbol{T}^{\pi}$ as a function that maps a given value function $\boldsymbol{V}$ into another value function $\boldsymbol{T}^{\pi}\boldsymbol{V}$ such that

$$\boldsymbol{T}^{\pi}\boldsymbol{V}(s) = \sum_{a\in\mathcal{A}} \pi(a|s)\left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a \boldsymbol{V}(s')\right], \qquad \forall s \in \mathcal{S}.$$

- The Bellman equation can be rewritten as $\boldsymbol{V}^{\pi} = \boldsymbol{T}^{\pi}\boldsymbol{V}^{\pi}$
- This operator is a γ-**contraction**, i.e. it makes value function closer by at least γ

$$\max_s |\boldsymbol{T}^{\pi}\boldsymbol{V}_1(s) - \boldsymbol{T}^{\pi}\boldsymbol{V}_2(s)| = \gamma \max_s \left|\sum_{a,s'} \pi(a|s)\mathcal{P}_{ss'}^a [\boldsymbol{V}_1(s') - \boldsymbol{V}_2(s')]\right|$$

$$\leq \gamma \max_s |\boldsymbol{V}_1(s) - \boldsymbol{V}_2(s)| \max_s \left|\sum_{a,s'} \pi(a|s)\mathcal{P}_{ss'}^a\right| = \gamma \max_s |\boldsymbol{V}_1(s) - \boldsymbol{V}_2(s)|$$

- Iterative Policy Evaluation: $\boldsymbol{V}_0 \rightarrow \boldsymbol{T}^{\pi}\boldsymbol{V}_0 \rightarrow \boldsymbol{T}^{\pi}\boldsymbol{T}^{\pi}\boldsymbol{V}_0 \rightarrow \cdots$

Appendix: Banach Fix Point Theorem

Theorem

Suppose $\boldsymbol{T}$ is a γ-contraction. Then under certain conditions,

- $\boldsymbol{T}$ *admits a* **unique** *fix point* $\boldsymbol{V}^*$*, i.e.* $\boldsymbol{T}\boldsymbol{V}^* = \boldsymbol{V}^*$*;*
- $\boldsymbol{V}^*$ *can be found as follows: define a sequence* $\{\boldsymbol{V_k}\}_{\boldsymbol{k}}$ *such that* $\boldsymbol{V_{k+1}} = \boldsymbol{T}\boldsymbol{V_k}$*. Then* $\boldsymbol{V}^* = \mathbf{lim}_{\boldsymbol{k}}\, \boldsymbol{V_k}$

- Proof can be found here
- $\boldsymbol{T}^{\boldsymbol{\pi}}$ is has a unique fix point
- $\boldsymbol{V}^{\boldsymbol{\pi}}$ is the fix point, according to the Bellman equation
- Iterative policy evaluation converges to $\boldsymbol{V}^{\boldsymbol{\pi}}$

Appendix: Bellman Optimality Operator

Definition

Define the Bellman Expectation Operator $\boldsymbol{T}$ as a function that maps a given value function $\boldsymbol{V}$ into another value function $\boldsymbol{TV}$ such that

$$\boldsymbol{TV}(s) = \max_{a \in \mathcal{A}} \left[\mathcal{R}_s^a + \gamma \sum_{s'} \mathcal{P}_{ss'}^a \boldsymbol{V}(s') \right], \qquad \forall s \in \mathcal{S}.$$

- The Bellman optimality equation can be rewritten as $\boldsymbol{V}^{\pi^{\text{opt}}} = \boldsymbol{TV}^{\pi^{\text{opt}}}$
- This operator is a γ-**contraction** as well

$$\begin{aligned}
\max_s \left| \boldsymbol{TV}_1(s) - \boldsymbol{TV}_2(s) \right| &= \gamma \max_{s,a} \left| \sum_{s'} \mathcal{P}_{ss'}^a [\boldsymbol{V}_1(s') - \boldsymbol{V}_2(s')] \right| \\
&\le \gamma \max_{s'} \left| \boldsymbol{V}_1(s') - \boldsymbol{V}_2(s') \right|
\end{aligned}$$

Appendix: Convergence of Dynamic Programming

- $\boldsymbol{T}$ has a unique fix point
- $\boldsymbol{V}^{\boldsymbol{\pi}^{\mathbf{opt}}}$ is the fix point, according to the Bellman optimality equation
- According to the Banach fix point theorem, **value iteration** converges to $\boldsymbol{V}^{\boldsymbol{\pi}^{\mathbf{opt}}}$
- **Policy iteration** (that integrates iterative policy evaluation & policy improvement) converges to $\boldsymbol{\pi}^{opt}$