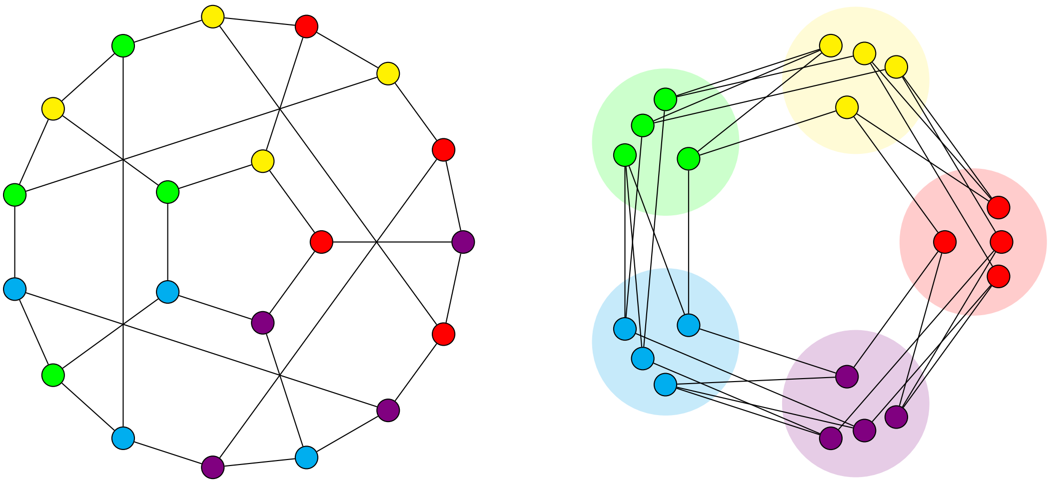
Master Thesis

Department of Mathematics

A discussion about the Pentagon Problem



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Abstract

The Pentagon Conjecture [33] states that every cubic graph of sufficiently high girth admits a homomorphism to the cycle of length 5.

In this thesis, we present some results related to the Pentagon Conjecture; we exploit the study of this problem to introduce some interesting methods such as the probabilistic method and local approaches, and to provide insights on many areas of combinatorics such as the study of the chromatic number.

The Pentagon Conjecture presents many similarities to some problems which naturally arise in other areas of combinatorics such as the study of gaps in the order $\prec$ restricted to cubic graphs. For this reason, in the first sections of this thesis, we analyse the techniques that have recently brought positive results for problems related to the Pentagon Conjecture. We then use the obtained tools to approach some approximations of the Pentagon Problem.

It is important to point out that our attention is mainly focused on expanding the reader's expertise on these topics and to present our attempt at working with the Pentagon Conjecture.

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Contents

1 Introduction

One interesting notion in graph theory is the concept of graph homomorphism. Indeed, it underlines the strong link between combinatorics and algebra, provides a generalisation of the well-known concept of colouring, and proves itself a useful tool in the study of many problems (edge reconstruction, density problem, etc.).

Definition 1.1. Let G and H be graphs. A *homomorphism* between G and H (sometimes called a H-colouring of G) is a map $\phi : V(G) \to V(H)$ such that if $xy \in E(G)$, then $\phi(x)\phi(y) \in E(H)$. We write $G \rightsquigarrow H$ to denote that there exists a homomorphism from G to H (and sometimes we say that G is H-colourable).

As its name suggests, this notion is an immediate generalisation of the concept of graph colouring. Indeed, observe that a graph G is k-colourable if and only if $G \rightsquigarrow K_k$; this follows by associating to each colour a distinct vertex of K_k.
Graph colouring received a lot of attention in graph theory, at least since the famous result of Brook [9]. In the generalised context of H-colourings, for some fixed graph H, we ask the nature of the set of graphs which admit a homomorphism to H.
Because of the equivalence between colourings and graph homomorphisms to complete graphs, and because the composition of graph homomorphisms is still a homomorphism, it is natural to try to extend results known for colourings to more general results about graph homomorphisms.
With this observation in mind, recall the aforementioned Brook's result.

Theorem 1.2 (Brook's Theorem, [9])**.** *Let G is a connected graph other than a cycle or a complete graph, then $\chi(G) \leq \Delta(G)$.*

In the language of graph homomorphisms, for $\Delta = 3$, this theorem gives that every cubic graph of girth at least 4 admit a homomorphism to $C_3 = K_3$. Nešetřil [33] asked whether a similar statement holds also for larger cycles.

Conjecture 1.3 (Original Pentagon Problem)**.** *For every $k \in \mathbb{N}$, there exists $g_k \in \mathbb{N}$ such that every cubic graph of girth at least g_k admits a homomorphism to C_{2k+1}.*

By Brook's Theorem, this conjecture holds for $k = 1$; recently, Hatami [17] modified a proof by Wanless and Wormald [40] to disprove the Pentagon Problem for $k = 3$ and therefore for every $k \geq 3$ (the first result in this sense was presented by Kostochka and Nešetřil [21]). Indeed, note that if $k < k'$, then $C_{2k'+1} \rightsquigarrow C_{2k+1}$. Therefore, the current state of the conjecture is as follows.

Conjecture 1.4 (Pentagon Problem)**.** *Every cubic graph of high enough girth admits a homomorphism to C_5.*

This conjecture also naturally arises when studying the density problem in the class of cubic graphs with respect to the order given by the definition of graph homomorphism. Slightly more in detail, we can define a quasiorder over the set of graphs as follows: $G \preceq H$ if and only if $G \rightsquigarrow H$. Moreover, if we restrict our attention to the family of the so-called *core graphs* (the quite rich family of graphs for which every automorphism is an isomorphism), the quasiorder that we defined is also a partial order. This quasiorder, and more generally the concept of graph homomorphism, is a useful tool for example in the study of edge reconstruction [26], [33].

A pair of graphs (G_1, G_2) with $G_1 \prec G_2$ is called a *gap*, if there is no graph G' such that $G_1 \prec G' \prec G_2$. The problem of finding all the gaps for some given (quasi)order is called the density problem, and it is a well-studied problem for many different orders [27]. This problem has been completely solved for the class of all graphs with respect to the quasiorder $\prec$ by Welzl [41] (the only two gaps are (K_0, K_1) and (K_1, K_2)), but it remains open for the class of cubic graphs. The Pentagon Problem arises when working to the Density Problem of cubic graphs.

We mention one last point of view from which the Pentagon Conjecture is a relevant problem (there are many more, for a reference see Nešetřil [33]). At least since the study of the chromatic number of planar graphs and the Four Colour Theorem, the study of colourings of minor-avoiding families is an active area in graph theory, and many results have been obtained about the colourability of sparse graphs (for a survey, Kostochka and Yancey [22]). The natural generalisation of this study to the language of graph homomorphisms has recently received a lot of attention (Nešetřil and Zhu [35] and Borodin et al. [8]). The Pentagon Problem is one example of conjecture about graph homomorphisms from a sparse family of graphs and it is of particular interest because we already know it does not hold for K_{2k+1} with $k \geq 3$ but it holds for K_3, therefore it is the last piece in the study of graph homomorphisms from high-girth cubic graphs (because high-girth cubic graphs are not generally bipartite, we have that they are not C_{2k}-colourable for any $k \in \mathbb{N}_+$).

With this examples in mind, in Section 4, we present another generalisation of the concept of colourings, from which we can obtain a nice result about colourability of minor-avoiding graphs of high girth.

The aim of this thesis is to present some results related to the Pentagon Problem; because the goal is not the proof of one single theorem, we present various point of view that we hope may help to understand the many facets of this problem.

Structure The thesis is organised as follows. In Section 2, we present a probabilistic proof that the Original Pentagon Conjecture does not hold, and we provide general tools to work with random regular graphs. In Section 3, we look at a positive result about the existence of homomorphisms from the family of cubic graphs of high girth to the Clebsch graph, which is equivalent to the existence of cut-continuous maps from said family to C_5. Following a similar idea, in Section 4 we present the circular chromatic number, we discuss how it is related to graph homomorphisms to cycles, and we explain in what sense the family of graphs avoiding a fixed minor is almost bipartite. In Section 5 we present our own reasoning about the Pentagon Problem (inspired from the positive result of Section 3), most importantly we introduce an approximation of the problem, and we work on a bound on the error of this approximation. One approach to the Problem raises a question about the chromatic number of triangle-free graphs which we analyse in Section 6. Finally, in Section 7, we present a generalisation of a recent result about 2-colourability of non-uniform hypergraphs, which we encountered studying almost-bipartite graphs (as in Section 4).

2 Construction of random regular graphs and homomorphisms to cycles

By Brook's Theorem, every cubic graph with girth at least 4 is 3-colourable. We can rewrite this result in the language of graph homomorphisms and state that cubic graphs of high enough girth are C_3-colourable. The Pentagon Problem asks whether it is possible to generalise this result. In its earliest presentation [33], this conjecture asked whether is it true that for every positive integer k, every cubic graph with high enough girth is C_{2k+1}-colourable. This conjecture was proved false by Kostochka and Nešetřil [21] for $k = 5$ (and therefore for every larger k). Since then, Wanless and Wormald [40] proved in 1999 that the statement does not hold for $k = 4$ and finally, Hatami [17] proved in 2006 that the conjecture fails also for $k = 3$. Currently, the Pentagon Conjecture reads as follows.

Conjecture 2.1 (Pentagon Conjecture)**.** *Every cubic graph with high enough girth is homomorphic to* C_5*.*

In this section, our goal is to prove the case $k = 6$ of the Pentagon Problem using probabilistic means. In particular, we focus on how we can apply the probabilistic method on the set of regular graphs.
In the first part of this section, we present a well-known model for random regular graphs (i.e. a probability space over the set of regular graphs with a given number of vertices) and we study how we can work with it; then we define the independence ratio of a (regular) graph and we show how to use this notion to prove the existence of a counterexample to the Pentagon Problem in the case $k = 6$.

2.1 Models of random regular graphs

In this section, all graphs are labelled graphs, unless it is explicitly stated otherwise; moreover, we assume that n and d are such that the set of d-regular graphs on n vertices is not empty.
The goal of this subsection is to analyse how we may apply the probabilistic method to study d-regular graphs on n vertices (for a deeper analysis, Wormald [42]). Let us denote with $\Omega_{n,d}$ the set of d-regular graphs on n vertices; to apply the probabilistic method, we have to study how to define a probability space over $\Omega_{n,d}$. The first probability measure that comes into mind is the uniform one. Let us denote with $\mathcal{G}_{n,d}$ the uniform probability space over $\Omega_{n,d}$. This probability space is easy to define and to understand (any result over this model has a natural interpretation), but it is not easy to work with it directly. For this reason, we introduce a different model for $\Omega_{n,d}$, and in particular we present $\mathcal{G}_{n,d}$ as the image probability of a random variable from an easier-to-study probability space.

2.1.1 Bollobás pairing model

Historically, the first model for random regular graphs is the pairing model, or configuration model, due to Bollobás [5]. While in its most general form this model provides a probability space over the graphs with a specific degree

sequence, we only present the particular instance of regular graphs as the generalised form does not provide any further insight.
The idea behind this model is to exploit the possibility to associate to each perfect matching over $n \cdot d$ vertices a d-regular multigraph by contracting together d-sets of vertices. A more formal construction is as follows.
Let $W_{n,d}$ be the union of n pairwise disjoint sets of vertices $W_1 \cup \ldots \cup W_n$, each of size exactly d; let ϕ be the map that associates to every graph P over $W_{n,d}$, the multigraph $\phi(P)$ over the vertex set $\{w_1, \ldots, w_n\}$ as follows: for any edge $xy \in P$ with $x \in W_i$ and $y \in W_j$ we define an edge $w_i w_j$ in $\phi(G)$ (note that this might cause loops or multiple edges, and therefore in general $\phi(G)$ is not simple). We say that $\phi(P)$ is obtained from P by contracting the sets W_i into distinct vertices.

Remark 2.2. If P is a perfect matching over $W_{n,d}$, then $\phi(P)$ is a d-regular multigraph. Because we want to study $\Omega_{n,d}$, let $S_{n,d}$ be $\phi^{-1}(\Omega_{n,d})$ which is the set of perfect matchings P over $W_{n,d}$ such that $\phi(P)$ is simple. We show how to obtain results over $S_{n,d}$ by studying the set of perfect matchings over $W_{n,d}$.

Let $\mathcal{P}_{n,d}$ be the uniform probability space over the set of perfect matchings over $W_{n,d}$; note that $S_{n,d}$ is a well defined event in $\mathcal{P}_{n,d}$ of positive probability (because of our assumptions on n and d). In order to define a random variable from $S_{n,d}$ to $\Omega_{n,d}$ let us denote with $\mathbb{P}_{S_{n,d}}$ the conditional probability of $\mathcal{P}_{n,d}$ with respect to $S_{n,d}$ ($\mathbb{P}_{S_{n,d}}$ is again a uniform probability).

Remark 2.3. It is of great interest for our application that $\mathcal{G}_{n,d} = \phi(\mathbb{P}_{S_{n,d}})$, which in our case is equivalent to say that every graph in $\Omega_{n,d}$ has a preimage of the same cardinality (because $\mathbb{P}_{S_{n,d}}$ and $\mathcal{G}_{n,d}$ are both uniform probability spaces). Let G in $\Omega_{n,d}$; then $\left|\phi^{-1}(G)\right| = (d!)^n$. Indeed, if P, Q are two perfect matchings obtained by permuting the vertices of $W_{n,d}$ in such a way that each W_i goes in itself, then $\phi(P) = \phi(Q)$; moreover, if this is not the case we have $\phi(P) \neq \phi(Q)$ (remember that we are working with labelled graphs), and there are exactly $(d!)^n$ of said permutations.

The following remark warns us that what we said is not yet enough to show that we can effectively use this model to study $\Omega_{n,d}$.

Remark 2.4. Suppose to have a sequence $(B_n)_{n\in\mathbb{N}}$ of probability spaces and a sequence of subspaces $(A_n)_{n\in\mathbb{N}}$ with $A_n \subseteq B_n$; moreover, let us assume that A_n has positive probability in B_n. This is not sufficient to translate an asymptotic result obtained in B_n to an asymptotic result in A_n. E.g. assume that P_n is a property that holds asymptotically almost surely in B_n (which means $\lim_{n\to\infty} \mathbb{P}_{B_n}[P_n] = 1$, from now on denoted as a.a.s.); it is possible that no element in A_n has the property P_n. As an example, take the case with $B_n = \mathcal{U}_{[0,1]}$ and $A_n = [0, 2^{-n}]$ and $P_n = (2^{-}n, 1]$, then P_n holds a.a.s. in B_n, but its intersection with A_n is always empty, even if A_n has always positive probability in B_n.
Observe that if we have $\lim_{n\to\infty} \mathbb{P}_{B_n}[A_n] > 0$, then this is not the case and any result holding a.a.s. on B_n holds also a.a.s. on A_n. We state this last remark as an independent lemma, because of its wide applicability (we do not present a proof, which can be obtained by contradiction).

Lemma 2.5. *Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of probability spaces, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets such that $A_n \subseteq B_n$ and $\mathbb{P}_{B_n}[A_n] > 0$. Let us denote with $\mathbb{P}_{A_n}$ the conditional probability of B_n with respect to A_n. If $\lim_{n\to\infty} \mathbb{P}_{B_n}[A_n] > 0$,* *then any sequence of events holding a.a.s. on B_n also holds a.a.s. with respect to $\mathbb{P}_{A_n}$.*

In our particular case, this means that we have to do some remarks before translating results obtained in $\mathcal{P}_{n,d}$ to results in $S_{n,d}$ and therefore in $\Omega_{n,d}$. In particular, in order to exploit our previous construction, we need to prove that, for fixed d, we have

$$\mathbb{P}_{\mathcal{P}_{n,d}}[S_{n,d}] \xrightarrow{n\to\infty} c_d > 0.$$

We prove the following statement.

Proposition 2.6. *Let d be a fixed positive integer. Then we have*

$$\mathbb{P}_{\mathcal{P}_{n,d}}[S_{n,d}] := \mathbb{P}[\phi(P) \text{ simple, with } P \in \mathcal{P}_{n,d}] \xrightarrow{n\to\infty} e^{-\frac{d^2-1}{4}}.$$

By Lemma 2.5, this proposition has an immediate corollary, which is of great interest for us.

Corollary 2.7. *If a sequence of events holds a.a.s. for $\mathcal{P}_{n,d}$, then its ϕ-corresponding sequence holds a.a.s. in $\mathcal{G}_{n,d}$.*

We need to conclude this subsection with a final remark. Remember that the Pentagon Problem is about d-regular graphs with high girth. In the spirit of the above remark, it could be the case that studying $\Omega_{n,d}$ is not enough to study d-regular graphs with high girth (it could be the case that these are rare in $\Omega_{n,d}$). This is not the case (see, [5] for a reference).

Proposition 2.8. *Let g and d be fixed. The proportion of pairings in $\mathcal{P}_{n,d}$ which yield graphs of girth at least g goes (for $n \to \infty$) asymptotically to*

$$\exp\left(-\sum_{k=1}^{g-1} \frac{(d-1)^k}{2k}\right).$$

The variation of this proposition that we need is as follows.

Corollary 2.9. *For g and d fixed, if a sequence of events holds a.a.s. for $\mathcal{P}_{n,d}$ then it also holds a.a.s. for graphs in $\mathcal{G}_{n,d}$ of girth at least g.*

2.2 The Poisson Paradigm and Brun's Sieve

In this subsection, we present two methods that have wide application in Combinatorics in general and can be applied to obtain some of the results of last subsection (Propositions 2.6, 2.8): the Poisson Paradigm and Brun's Sieve. For a deeper analysis of the Poisson Paradigm, we refer to the Alon and Spencer book [3, Chapter 8] (everything that we present here about these methods can be found there).

It is well known that if we have a sequence of random variables X_n such that X_n is the sum of n i.i.d. Bernoulli variables of parameter p_n, and if $\lim_{n\to\infty} np_n = \lambda$, then the distribution of X_n tends to the distribution of a Poisson random variable of parameter λ. This result is called Poisson Limit Theorem.

The Poisson Paradigm allows us to obtain similar results even when the random variables that constitute X_n are not i.i.d., but are "mostly independent" in some

sense. In particular, many results follow from the fact that, under hypothesis to be specified, we can say that $\mathbb{P}[X_n = 0] \simeq e^{-\mathbb{E}[X_n]}$ for n large enough.
Poisson Paradigm is not the name of a specific theorem but is the name of the approach in which we try to write a sequence of combinatorial random variables as the sum of almost independent indicator random variables, and we try to deduce the behaviour of the limit.
The most widely applied use of the Poisson Paradigm is called Brun's Sieve.

2.2.1 Brun's Sieve

Let $(\Omega_n, \mathbb{P}_n)_{n\in\mathbb{N}}$ be probability spaces; for each n, let $B_1, \ldots, B_{m(n)}$ be events in Ω_n, and let $X_n = \sum_{i=1}^{m(n)} \mathbb{1}_{B_i}$ be the random variable that counts the number of B_i that hold. Moreover, for any given positive integer r, let

$$S_n^{(r)} = \sum_{\{i_1 \neq \ldots \neq i_r\} \subseteq [m(n)]} \mathbb{P}_n[B_{i_1} \wedge \ldots \wedge B_{i_r}],$$

be the sum over all subsets A of $[m(n)]$ of size r of the probability that each event B_i with $i \in A$ holds simultaneously. Finally, let

$$X_n^{(r)} = X_n(X_n - 1) \ldots (X_n - r + 1).$$

Theorem 2.10 (Brun's Sieve, [3])**.** *Suppose that, in the context just exposed, there exists a constant λ such that $\mathbb{E}[X_n] = S^{(1)} \xrightarrow{n\to\infty} \lambda$ and such that, for every fixed positive integer r,*

$$\mathbb{E}\left[\frac{X_n^{(r)}}{r!}\right] = S_n^{(r)} \xrightarrow{n\to\infty} \frac{\lambda^r}{r!}.$$

Then we have $\mathbb{P}_n[X_n = 0] \xrightarrow{n\to\infty} e^{-\lambda}$ and, more generally,

$$\mathbb{P}_n[X_n = t] \xrightarrow{n\to\infty} \frac{\lambda^t}{t!} e^{-\lambda}.$$

2.2.2 Some applications

We now give an idea on how to prove some propositions stated previously in this section using Brun's Sieve.

- Proposition 2.6. Recall the construction procedure for the pairing model $\mathcal{P}_{n,d}$. For two vertices $x \in W_i$ and $y \in W_j$, let $B_{x,y}^{i,j}$ be the event of the graphs in $\mathcal{P}_{n,d}$ containing the edge xy. For any distinct $i, j \in [n]$, we define the random variables $Z_i = \sum_{x\neq y; x,y\in W_i} \mathbb{1}_{B_{x,y}^{i,i}}$, which counts the loops in the i-th vertex, and $Z_{i,j} = \sum_{x_i\neq y_i\in W_i, x_j\neq y_j\in W_j} \mathbb{1}_{B_{x_i,x_j}^{i,j}} \mathbb{1}_{B_{y_i,y_j}^{i,j}}$, which counts the number of pairs of multiple edges between W_i and W_j. We also define,

  $$X_n = \sum_{i\in[n]} Z_i, \qquad Y_n = \sum_{1\leq i<j\leq n} Z_{i,j}.$$

  Note that $Y_n(P) = 0$ if and only if $\phi(P)$ has no multiple edge; and $X_n(P) = 0$ if and only if $\phi(P)$ has no loop. Moreover, both these random variables are nonnegative.

We want to apply Brun's Sieve Theorem; in particular, we are interested in finding $\mathbb{P}[X_n + Y_n = 0]$ which, for what we just observed, equals the probability that $P \in \mathcal{P}_{n,d}$ is such that $\phi(P)$ is a simple graph (i.e. $\mathbb{P}[S_{n,d}]$). We just show how to obtain $\mathbb{E}[X_n + Y_n]$ and that the result is coherent with Proposition 2.6.

- Calculate $\mathbb{E}[X_n]$. Let us fix $i \in [n]$, and $x, y \in W_i$ two distinct vertices. The probability of the event $\{P \in \mathcal{P}_{n,d} : xy \in E(P)\}$ is $\frac{1}{nd-1}$ (it holds more generally for any two fixed vertices in $W_{n,d}$). Observe that we can select a couple of vertices in W_i in $\binom{d}{2}$ distinct ways. By linearity of expectation we get $\mathbb{E}[Z_i] = \binom{d}{2}\frac{1}{nd-1}$ and also

$$\mathbb{E}[X_n] = n \cdot \binom{d}{2}\frac{1}{(nd-1)} \simeq \frac{d-1}{2}.$$

- Calculate $\mathbb{E}[Y_n]$. Let $i \neq j$; given two pairs (u_i, u_j) and (v_i, v_j) such that $u_i, v_i \in W_i$ and $u_j, v_j \in W_j$, we have that the probability that both $u_i u_j$ and $v_i v_j$ are edges is exactly $1/(nd-1)(nd-3)$. We can choose these pairs in $2\binom{d}{2}^2$ ways, and we can choose i and j in $\binom{n}{2}$ ways. Therefore we have:

$$\mathbb{E}[Y_n] = 2\binom{n}{2}\binom{d}{2}^2 \frac{1}{(nd-1)(nd-3)} \simeq \frac{(d-1)^2}{4}.$$

We should then proceed by calculating the factorial moments of the random variable $X_n + Y_n$ (for explicit calculations, see the original article in which Bollobás introduced the pairing model [5]).

- Proposition 2.8. In this example, we work over $2n$ vertices. An ℓ-cycle for a configuration $P \in \mathcal{P}_{2n,d}$ is a set of ℓ edges of P such that the corresponding edges in $\phi(P)$ form a cycle. For some given ℓ-cycle C over $W_{2n,d}$ (set formed by ℓ pairs of vertices –edges– which form an ℓ-cycle in the image with respect to ϕ), let B_C be the set of matchings in $\mathcal{P}_{2n,d}$ that contain these edges. To apply Brun's Sieve, we need to study the expectation of $Z_\ell(P) = \sum_C \mathbb{1}_{B_C}(P)$, the random variable which counts the ℓ-cycles of P.

  Let us define the function $N(k) = \frac{\binom{2k}{2}\binom{2k-2}{2}\cdot\ldots\cdot\binom{2}{2}}{k!}$ that counts the configurations over $2k$ vertices, and let C be a given ℓ-cycle over $W_{2n,d}$; there are exactly $N(dn-\ell)$ configurations that contain C (indeed, once determined those ℓ edges, we have to determine just $dn-\ell$ others). Moreover, there are $\frac{(2n)(2n-1)\ldots(2n-\ell)}{2\ell}$ cycles of length ℓ over the $2n$ vertices $\{w_1, \ldots, w_{2n}\}$. Let C' be any one of those; then there are $(d(d-1))^\ell$ cycles in $W_{2n,d}$ that are mapped into C' by ϕ.

  Therefore, we have,

$$\begin{aligned}\mathbb{E}[Z_\ell] &= \frac{1}{N(dn)}\frac{(2n)(2n-1)\ldots(2n-\ell)}{2\ell}(d(d-1))^\ell N(dn-\ell)\\ &\xrightarrow{n\to\infty} \frac{(d-1)^\ell}{2\ell}.\end{aligned}$$

From which it follows the expectation of the random variable which counts the cycles of length at most g. For the details of the complete proof, see Bollobás [5].

2.3 A first use of the independence ratio

In this subsection, we present a first approach to the study of the existence of homomorphisms to odd cycles. In particular, we define the independence ratio of a graph, which gives an immediate upper bound to the length of the cycles this graph can be mapped into. We also present a result due to Bollobás [6] and one of its refinements due to McKay [29] which allow us to obtain explicit results in the spirit of the original Pentagon Conjecture.

2.3.1 The independence ratio of regular graphs

Definition 2.11. Let G be a graph, we denote with $\alpha(G)$ its independence number. The *independence ratio* $i(G)$ of G is defined as $i(G) = \frac{\alpha(G)}{|V(G)|}$.

The importance of this definition in our context derives from a Lemma found in Albertson and Collins [1] that we prove just for odd cycles.

Lemma 2.12. *Let H be a vertex-transitive graph and G a graph such that $G \rightsquigarrow H$, then $i(G) \geq i(H)$.*

Proof. As we said, we assume $H = C_{2k+1}$ (over the vertex set $\{v_0, \ldots, v_{2k}\}$ with the natural edges); note that $i(H) = \frac{k}{2k+1}$. If we let $n = |V(G)|$, then without loss of generality we have $\left|f^{-1}(v_0)\right| \leq \frac{n}{2k+1}$; let us denote $U = f^{-1}(\{v_1, v_3, \ldots, v_{2k-1}\})$ and $V = f^{-1}(\{v_2, v_4, \ldots, v_{2k}\})$. Because f is an homomorphism, both U and V are independent sets. Because we have $|U| + |V| \geq n - \frac{n}{2k+1}$, at least one between U and V has to have cardinality at least $\frac{kn}{2k+1}$ (without loss of generality, it is U). Hence, the independence ratio of G is at least $\frac{|U|}{n} \geq \frac{k}{2k+1}$. □

Therefore, if the Pentagon Conjecture were true in its original form, the independence ratio of every random cubic graph of high enough girth would be bigger than any number arbitrarily near to $\frac{1}{2}$. As the following theorem shows, this is not the case.

Theorem 2.13 (McKay, [29]). *There are cubic graphs of arbitrarily high girth with independence ratio less than* 0.4554. *Something stronger holds: the sequence of events A_n in $\mathcal{G}_{n,3}$ of cubic graphs with independence ratio smaller than* 0.4554 *holds a.a.s.*

There is an immediate corollary of these two results.

Corollary 2.14. *There are cubic graphs of arbitrarily high girth without homomorphisms to C_{13}.*

We now prove an earlier version of Theorem 2.13, due to Bollobás [6] (which is enough to prove the above corollary for C_{2k+1} for some larger k). The sharpened bound stated above, due to McKay [29], can be obtained using very similar methods but a more sophisticated analysis.

Theorem 2.15 (Bollobás [6]). *There exists a function $f : \mathbb{N}_{\geq 3} \to (0,1)$ with $f(d) \leq \frac{4\log(d)}{d}$ such that for any natural integer $d \geq 3$ a.a.s. every d-regular graph G in $\mathcal{P}_{n,d}$ is such that*

$$i(G) \leq \frac{f(d)}{2}.$$

Proof. We use the pairing model to show that for d fixed and $2n$ large, almost every d-regular graphs on $2n$ vertices has independent number at most $2f(d)n - 2\sqrt{n}$ for some function f yet to be determined.
Recall that we denoted with $N(dn)$ the number of possible configurations over $2dn$ vertices:

$$N(dn) = \frac{\binom{2dn}{2}\binom{2dn-2}{2}\cdots\binom{2}{2}}{(dn)!} = \frac{(2dn)!}{2^{dn}(dn)!}.$$

Let $s = s(n) = \lfloor 2f(d)n - \sqrt{2n} \rfloor$ and $\beta(n)$ such that $s(n) = \beta(n)n$ (observe that $\beta(n) \to f(d)$); for $P \in \mathcal{P}_{2n,d}$, we denote with $S(P)$ the number of $ds(n)$-subsets of $W_{2n,d}$ of the form $U = W_{i_1} \cup \ldots \cup W_{i_s}$ such that U spans no edge. We compute the expectation for S.
If U spans no edge in P, all the sd vertices of U are paired with one of the $2nd - sd$ vertices not in U. This can be done in $(2nd-sd)(2nd-sd-1)\ldots(2nd-2sd+1)$ ways. Moreover, the remaining vertices can be paired in $N(d(n-s))$ ways. Therefore, the number R of configurations for which U spans no edge is:

$$R = (2nd - sd)(2nd - sd - 1)\ldots(2nd - 2sd + 1)N(d(n-s)).$$

Also, we can choose U in $\binom{2n}{s}$ ways (the value of R does not depend on U but on its size, which we fixed). Therefore by linearity of expectation we have:

$$\mathbb{E}[S] = \binom{2n}{s} \cdot \frac{R}{N(dn)} = \frac{(2n)!}{s!(2n-s)!}\frac{((2n-s)d)!}{((n-s)d)!}\frac{(nd)!}{(2nd)!}2^{sd}.$$

By applying Stirling's Formula for factorial numbers, we obtain

$$\mathbb{E}[S] < Cn^{-1/2}Y(n, f(d)),$$

where Y is a function that is strictly smaller than 1 if n is large enough and f is small enough. We can take f to be $f(d) = 4\frac{\log(d)}{d}$ (which works well for d larger than 20); but in particular, we can choose $f(3) = 12/13$. The explicit results of the calculations are as follows.

$$\mathbb{E}[S] < Cn^{-1/2}2^{-((2-\beta)d-2)n}\beta^{-\beta n}(1-\beta)^{(\beta-1)dn}(2-\beta)^{(2-\beta)(d-1)n}.$$

For f as mentioned above, we have that $\mathbb{E}[S] < Cn^{-1/2}$ and therefore it holds $\mathbb{P}[S > 1] \xrightarrow{n\to\infty} 0$. □

We explicitly enunciate the version of this theorem for graphs with high girth, even if it is implied by Proposition 2.8.

Corollary 2.16. *Let $i(d,g)$ be the infimum over the independence ratio of every d-regular graph of girth at least g. We have $i(d,g) \leq \frac{f(d)}{2}$ for every g.*

Proof. Let us fix g and d; by the proof of Proposition 2.8, we have that almost every configuration in $\mathcal{P}_{n,d}$ has fewer than $\log(n)$ cycles of length smaller than g and contains at most $\beta n = \lfloor f(d)n - \sqrt{n} \rfloor$ independent vertices. Let G_0 be one of these graphs and let G_1 be a graph obtained from G_0 by removing an edge from each cycle of length at most g. We have that G_1 has independent number at most $\lfloor f(d)n - \sqrt{n} \rfloor + \log(n) < f(d)n$. Finally, by joining together two or more copies of G_1 so that the resulting graph is d-regular, we obtain the desired graph. □

3 High-girth cubic graphs are homomorphic to the Clebsch graph

In the previous section, we showed one probabilistic proof that general high-girth cubic graphs do not map to C_{13}. In this section, we present one positive homomorphism result due to Devos and Šámal [10]; our goal is to study a method to prove the existence of homomorphisms from a family to a specific graph.
More in detail, we show that if G is a (sub)cubic graph of girth at least 17, then there exists a homomorphism from G to the Clebsch graph PQ_4 (the 4-dimensional projective cube, explicit definition follows). This result is interesting also because homomorphisms to PQ_{2k} have been the object of multiple studies and conjectures (see Naserasr [32], Seymour [38]). Also relevant for our approach is the fact that homomorphisms to PQ_{2k} can be analysed by studying the existence of pairwise disjoint cut complements; problems strictly linked to the latter have received great attention (see Bondy and Locke [7], Zýka [44]).
In the first subsection, we present some notations and definitions, and we underline the link between the existence of pairwise disjoint cut complements and homomorphisms to projective cubes. In the last subsection, we present the main theorem and we prove it.

3.1 Cut complements and homomorphisms to projective cubes

The key to the success of Devos and Šámal's approach is the fact that homomorphisms to PQ_4 can be studied locally. This is possible by an equivalence between the existence of homomorphisms from G to PQ_4 and cut complements. We need to introduce some notation before studying this equivalence.

Definition 3.1 (Cuts and cut complements)**.** Let G be a graph; we call *cut* a set of edges of the form $\delta(U) = \{xy \in E(G) : x \in U, y \notin U\}$ for some $U \subseteq V(G)$. A set of edges of the form $C = E(G) \setminus \delta(U)$ is called a *cut complement.* Equivalently, $C \subseteq E(G)$ is a cut complement if there exists $U \subseteq V(G)$ such that $C = E(G|_U \cup G|_{V(G)\setminus U})$.
For a given graph G, it is an interesting and well-studied problem to determine the maximum size of a cut in G; this value is denoted by MAXCUT(G) (among the people that studied this problem, Bondy and Locke [7], Zýka [44]).
Finally, a map $f : E(G) \to E(H)$ is called *cut-continuous* if the preimage of every cut in H is a cut in G.

The study of MAXCUT(G) is equivalent to the study of minimal cut complements.

Definition 3.2 (Cube graph)**.** Let $\{0,1\}^n$ be the set of n-tuples of elements in the set $\{0,1\}$. We call *n-dimensional cube graph* the graph Q_n over the vertex set $\{0,1\}^n$ in which two vectors are adjacent if and only if they differ in exactly one coordinate. We can notice that, in particular, Q_n is an n-regular graph over 2^n vectors.

A way of defining new graphs from some given graph is by contraction. Given a graph G and $x, y \in V(G)$, the graph obtained from G by contracting x and y

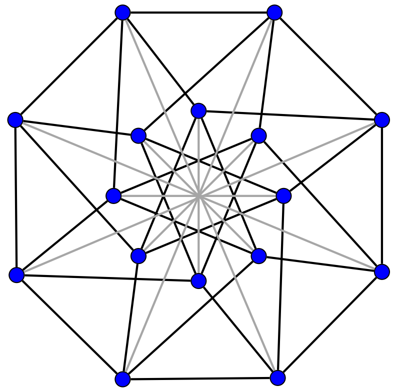

Figure 1: The Clebsch graph PQ_4.

is the graph over $V(G) \setminus \{x, y\} \cup \{*\}$ such that for all z in $V(G) \setminus \{x, y\}$ we have that z is adjacent to $*$ in the contracted graph if and only if z is adjacent in G to at least one between x and y. This definition can be extended to a family of pairs of vertices or a family of sets of vertices. It is now time to define the projective cubes PQ_n.

Definition 3.3 (Projective cube graph)**.** We define the *n-dimensional projective cube* PQ_n, as the graph obtained from Q_{n+1} by contracting all the pairs of antipodal vertices. It is not difficult to observe that PQ_n is an $n+1$-regular graph over 2^n vertices.

The study of minimal cut complements can be linked with the problem of finding many disjoint cut complements. This latter problem is strictly linked with the study of homomorphisms to PQ_{2k}. In particular, it holds the following equivalence.

Proposition 3.4. *Let G be a graph and k a positive integer. The following are equivalent.*

a) There are $2k$ pairwise disjoint cut complements,

b) There exists a homomorphism between G and PQ_{2k},

c) There exists a cut-continuous mapping between $E(G)$ and $E(C_{2k+1})$.

Before proceeding to the proof of this proposition, let us just point out that there is a strong relation between cut-continuous maps and homomorphisms.

Remark 3.5. Every graph homomorphism h from $V(G)$ to $V(H)$ naturally generates a cut-continuous mapping ϕ, which is given by $\phi(xy) = h(x)h(y)$. Therefore, the concept of graph homomorphism is stronger than the one of cut-continuous mapping. However, it has been shown that the existence of cut-continuous maps often implies the existence of homomorphisms between graphs (Nešetřil and Šámal [34]).
This provides yet another link between the result presented in this section and the Pentagon Conjecture. Even if it does not seem likely that the methods used in [34] can be applied in this case, the strong bond between homomorphisms to

PQ_4 and cut-continuous maps is interesting to examine. Indeed, we prove that cubic graphs of high girth admit homomorphisms to PQ_4 and hence they admit cut-continuous maps to C_5.

Proof of Proposition 3.4. We show that $a) \implies b) \implies c) \implies a)$.
Firstly, let us denote with H_{2k+1} the graph over $\{0,1\}^{2k+1}$ with any two vertices x, y adjacent if and only if they coincide in just one coordinate. We may notice that H_{2k+1} has two connected components (vectors with even or odd number of $0's$) both isomorphic to PQ_{2k} (they are isomorphic among themselves by symmetry, i.e. by sending each vector to its complementary; and they are both isomorphic with PQ_{2k} by taking the identity map in Q_{2k+1}).

$a) \to b)$ We show that if $a)$ holds, then G admits a homomorphism to H_{2k+1}, which suffices.

Let $S_1, \ldots, S_{2k}$ be the pairwise disjoint cut complements, and $S_{2k+1} = E(G) \setminus \bigcup S_i$ their complement. Let also $U_i \subseteq V(G)$ be such that S_i is the complement of the cut induced by U_i. Note that $S_1, \ldots, S_{2k+1}$ form a partition of $E(G)$ (and that also S_{2k+1} is a cut complement, by taking the symmetric difference of all the U_i).

Let γ the map that sends a vertex v to the vector $x_v \in \{0,1\}^{2k+1}$ such that $(x_v)_i = \mathbb{1}_{v \in U_i}$. Then γ is a homomorphism to H_{2k+1} because if two vertices are adjacent, the edge which connects them is in exactly one of the S_i.

$b) \to c)$ Since being cut-continuous is closed with respect to functions composition, it suffices to show that H_{2k+1} admits a cut-continuous map to C_{2k+1}. Let us denote $E(C_{2k+1}) = \{e_1, \ldots, e_{2k+1}\}$ and let g be the map $g : E(H_{2k+1}) \to E(C_{2k+1})$ that sends the edge xy into the edge e_i if x and y agree exactly on the i-th coordinate. By taking the preimage of any cut R in C_{2k+1}, which has even cardinality, we obtain our claim.

$c) \to a)$ Let f be the homomorphism and $x_i x_{i+1}$ an edge in C_{2k+1}; let $U = f^{-1}(x_i) \cup f^{-1}(x_{i+1})$. Then $f^{-1}(x_i x_{i+1}) = E(G) \setminus \delta(U)$. This implies that $E(G)$ admits $2k+1$ disjoint cut complements.

□

Finally, before starting to examine the main result of this section, we point out that the existence of homomorphisms to PQ_{2k} is also the topic of various conjectures, one of which states.

Conjecture 3.6 (Seymour [38])**.** *Every planar graph with all odd cycles of length at least $2k+1$ has a homomorphism to PQ_{2k}.*

Because in the case $k = 1$ we have that PQ_{2k} is isomorphic to K_4, and because the existence of a homomorphism from G to K_i is equivalent to i-vertex colourability, we have that this conjecture is a generalisation of the well known Four Colour Theorem.

3.2 Homomorphisms to the Clebsch graph

We now present the main result of the aforementioned article by DeVos and Šámal, [10].

Theorem 3.7. *Every cubic graph of girth at least* 17 *admits a homomorphism to* PQ_4.

We need some technical definitions and remarks before introducing the proof of the theorem (the definitions we give are useful in the context of this proof, but are not universally valid and may create confusion outside this context).

Definition 3.8 (labellings and weights)**.** Let G be a graph. We define a *labelling* of G as a 4-tuple $X = (X_1, \ldots, X_4)$ such that $X_i \subseteq E(G)$. We call X a *cut labelling* (resp. *cut complement labelling*) if every X_i is a cut set (resp. a cut complement). In general, we call X *wonderful* if for every $i \neq j$ we have $X_i \cap X_j = \emptyset$.
Let X be a labelling of G and e be an edge in $E(G)$, we call $\{i : e \in X_i\}$ the *label* $\ell_X(e)$ of e with respect to X. The *weight* of e in X is defined as $w_X(e) = |\ell_X(e)|$. Finally, we define the *cost* of every single edge as $\text{cost}_X(e) = \alpha(w_X(e))$, where α is defined by $\alpha(0) = 0,\ \alpha(1) = 1,\ \alpha(2) = 10,\ \alpha(3) = 40$ and $\alpha(4) = 1000$; in the same way, $\text{cost}(X) = \sum_e \text{cost}_X(e)$.

Because of the equivalence in Proposition 3.4, to prove that G has a homomorphism to PQ_4 it suffices to prove that G admits four pairwise disjoint cut complements. By our last definition, if we can prove that G has a wonderful cut complement labelling, we have the existence of our homomorphism.
Therefore, to prove the theorem we show that under our assumptions on the regularity and girth of G every cut complement labelling of minimal cost is wonderful. In particular, we proceed locally; i.e. we show that if a cut complement labelling is not wonderful, it is possible to modify it locally (near an edge that is contained in two of the sets of the labelling) to obtain a labelling with lower cost. To do so, we use the fact that every cubic graph of high girth is locally similar to a cubic tree. Indeed, we describe an operation that allows us to obtain a labelling with a smaller cost in fixed cubic trees with central edge of weight bigger than one, and we then apply this operation locally in our cubic graph.
We need to introduce some notation about cubic graphs and to define the main operation that allows us to change the labellings.

Definition 3.9 (Rooted trees and internal vertices)**.** We define recursively the *cubic rooted tree* T_i as follows. T_1 is an edge with one rooted vertex. Given T_i, we construct T_{i+1} as follows. Let (U_1, x_1) and (U_2, x_2) be two copies of T_i, and let (U, x) be the graph obtained by contracting to x the roots x_1 and x_2 in the union graph of U_1 and U_2. Then we define T_{i+1} over $U \sqcup \{x'\}$ with root x' and with edge set the edges of U with also an edge between x and x'. We call the unique edge adjacent to the root the *root edge* of the rooted tree T_i.
Moreover, we define $2T_i$ to be the graph obtained by identifying in opposite directions the rooted edges of two disjoint copies of T_i.
We call a vertex in T_i or $2T_i$ *interior* if it is the root or if it is not a leaf.
A cut is called *internal* if it can be written as $\delta(U)$ with U a set of interior vertices. A cut labelling is internal if every component that it has is an internal cut.

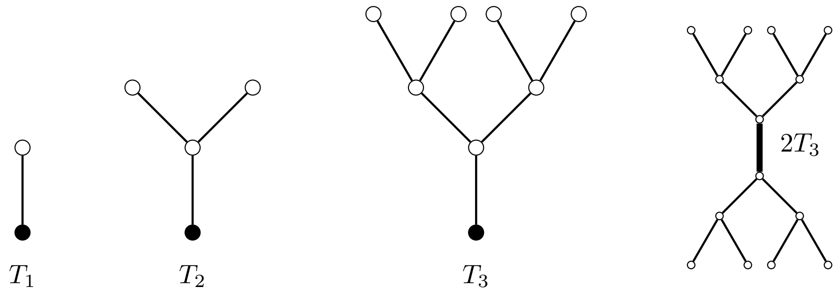

Figure 2: Cubic rooted trees.

We are now ready for a remark.

Remark 3.10. If $C = \delta(U)$ is a cut and $D = E(G) \setminus \delta(V)$ is a cut complement, then the symmetric difference $C\Delta D$ is a cut complement given by

$$\begin{aligned} C\Delta D &= (C \setminus D) \cup (D \setminus C) = E(G) \setminus (\delta(U)\Delta\delta(V)) \\ &= E(G) \setminus \delta(U\Delta V). \end{aligned}$$

In particular, if X is a cut complement labelling and Y is a cut labelling, we have that $X\Delta Y$ (the componentwise operation) is a cut complement labelling.

We can now proceed to analyse the first piece needed for the theorem.

Lemma 3.11. *Let X be a cut complement labelling of $2T_2$ of minimal cost. Then the weight of the central edge is at most 2.*

Proof. Let X be a cut complement labelling of $2T_2$ with weight of the central edge bigger than 2; we find an internal cut labelling Y such that $\mathrm{cost}(X\Delta Y) < \mathrm{cost}(Y)$ (note that the fact that Y is internal is pivotal to the proof of our theorem; indeed, this allows to extend such a result to every graph locally isomorphic to $2T_2$).
We use the notation represented in the following figure (where x is either one of the vertices of the central edge e):

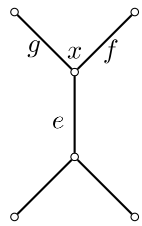

Figure 3: We define the edges e, f, g and the vertex x.

Moreover, for $I \subseteq \{1, \dots, 4\}$, let us denote with Y_I the cut labelling of $2T_2$ that has $\delta(\{x\}) = \{e, f, g\}$ in the coordinates corresponding to I and $\emptyset$ in the other positions.
If $I = \ell_X(e) \cap \ell_X(f) \cap \ell_X(g)$ is not empty, it means that there is at least one index (let it be the index 1) for which e, f, g belong to X_1. In this case, we can take $Y_{\{1\}} = (\{e, f, g\}, \emptyset, \emptyset, \emptyset)$; it is immediate to notice that $Y_{\{1\}}\Delta X$ has cost

strictly less than X (because only the first coordinate of X changes, and its weight is strictly less) and it is a cut complement labelling by our last remark. Therefore we may assume that the intersection $\ell_X(e) \cap \ell_X(f) \cap \ell_X(g)$ is empty. We are in one of the following cases.

a) $w_X(e) = 4$ (which means e is in every coordinate of X). We have two possible cases. Either $\ell_X(f) = \ell_X(g) = \emptyset$, which means that neither f nor g are in any coordinate of X in which case we can take $Y_{\{1\}}$ and reason similarly to above; or $\ell_X(f) \cup \ell_X(g) = I \neq \emptyset$. In such a case, we can directly take Y_I (because the intersection of all three indices sets is empty, we are exchanging the place of f and g in the coordinates, and reducing the contribution of e).

b) If e is in three coordinates (it cannot be just in two because else we would already have our thesis), we may assume $\ell_X(e) = \{1, 2, 3\}$ and $w_X(e) \geq w_X(f) \geq w_X(g)$; moreover, $w_X(g) \leq 2$ because otherwise they would have nontrivial intersection. In a first scenario, we have $\ell_X(f) \cap \ell_X(e) = I \neq \emptyset$, in which case, following what we did before we have a reduction of the cost taking $Y_I \Delta X$. In the second case, both $\ell_X(f)$ and $\ell_X(f)$ are subsets of $\{4\}$; then we are done by taking, again, $Y_{\{1\}}$ (indeed the cost for e, f, g passes from at least $\alpha(3)$ to at most $3\alpha(2)$).

□

We state now a more technical lemma, of the same nature as the last one, from which Theorem 3.7 follows. We do not present a proof for this lemma because the one presented in [10] heavily relies on computer computations.

Lemma 3.12. *Let X be a cut complement labelling of $2T_9$ in which every edge has weight at most 2 and for which the weight of the central edge is exactly 2. There exists an internal cut labelling Y for which cost$(X\Delta Y) <$ cost(X).*

A first computational approach to this problem could be to find, for every cut complement labelling X which is not wonderful, an internal cut labelling Y as in the statement. But it is not difficult to show that the possible number of labellings is out of the current computational possibilities. Therefore, it is necessary to focus on some particular, worst-case scenarios that are obtained by defining a partial order on some auxiliary structure. For a complete analysis of the algorithm, see [10, Section 2]
We are now ready to prove the main result of this section.

Proof of Theorem 3.7. By Proposition 3.4, applied for $k = 2$, it suffices to prove that every cubic graph with girth at least 17 admits a wonderful cut complement labelling.
Let G be a cubic graph of girth at least 17 and let X be a cut complement labelling of minimum cost. By Lemma 3.11, we can assume that every edge has weight at most 2 with respect to X (indeed, if it is not the case, we can find a local cut labelling Y so that $Y\Delta X$ is a cut complement labelling of strictly lower cost). Now, assume there is one edge e of weight exactly 2; then we can take the restriction of X to the subgraph of G internally isomorphic to $2T_9$ that has e as its central edge (it is uniquely defined, but it might have some contracted leaves). By Lemma 3.12, we can find a cut labelling Y such that $X\Delta Y$ has cost strictly less than the cost of X, and obtain the absurd.

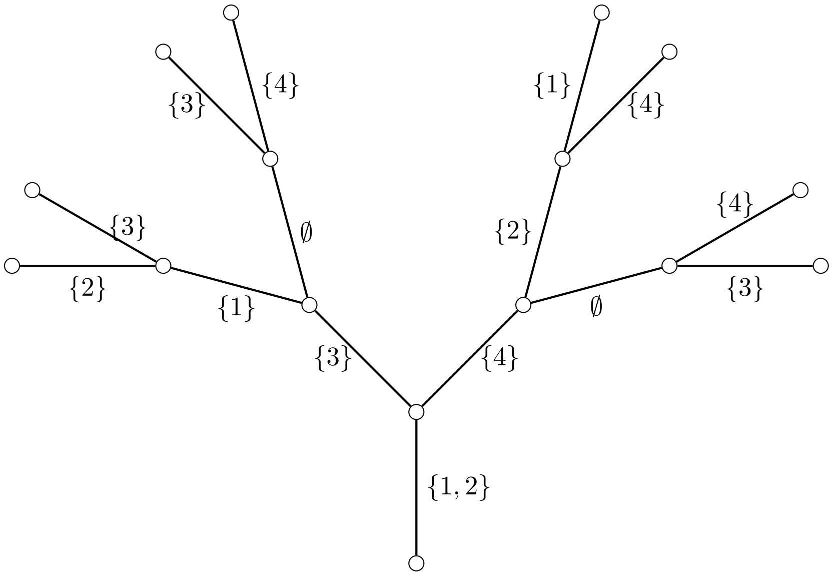

Figure 4: A difficult labelling of T_4.

Therefore if X has minimal cost, then it is wonderful; and because the cost map is discrete, there exists at least one cut complement labelling with minimal cost. □

There is one quick generalisation of Theorem 3.7.

Corollary 3.13. *Every graph with maximum degree* 3 *and girth at least* 17 *is homomorphic to* PQ_4.

Proof. Let G be our subcubic graph, and let us define $r = \sum_v (3 - \deg(v))$ the measure of how G fails to be 3-regular.
Let H be an r-regular graph of girth at least 17 (which exists by Proposition 2.6); moreover, let the graph G_0 be constructed as follows. Take $|V(H)|$ copies of G indexed by the set $V(H)$; iteratively for every edge uv in H, take one vertex x of degree less than three in the copy of G indexed by u and one vertex y of degree less than three in the copy of G indexed by v; add to G_0 the edge xy.
By repeating the construction, we obtain a graph G_0 which is a 3 regular graph with girth at least 17 and with subgraphs isomorphic to G. From Theorem 3.7 we get that G_0 admits a homomorphism to C_5 and therefore so does G. □

Remark 3.14. Before concluding this section, it is interesting to point out that it is not a coincidence the fact that we imposed $\alpha(1) \neq 0$. Indeed, one could wonder why $\alpha(1) \neq 0$ if we just want to limit the number of edges that are in at least two component of our labelling.
The idea behind the reason for which this precaution is necessary is that there are some difficult examples in which an easier cost function does not work. Indeed, if we set $\alpha(1) = 0$ there are some cuts complements which are not

wonderful and such that it is not possible to reduce their cost with the aforementioned operation in just one step. The idea is that setting $\alpha(1) = 1$ helps us decide what to do in some particular situations.
One example is as follows. If we set $\alpha(1) = 0$ and we have a cut complement labelling X in which every edge has weight at most 2 and exactly one edge e has weight 2, then the algorithm we presented fails if there is no way of reducing the weight of e without increasing some the weight of some other vertex. For an explicit example, Figure 4.

Finally, we present a quick example of the reason for which a similar method cannot be trivially applied to solve the Pentagon Conjecture.

Remark 3.15. For every $k \geq 3$, there exists a map $h : V(2T_k) \to V(C_5)$ such that every modification h' of h which preserves the value of h on the leaves is not a homomorphism. Indeed, for any fixed k, we have that there exists a copy of T_3 inside $2T_k$ in which leaves of $2T_k$ corresponds to leaves of T_3. Consider any mapping h that takes on those leaves values as in Figure 5.

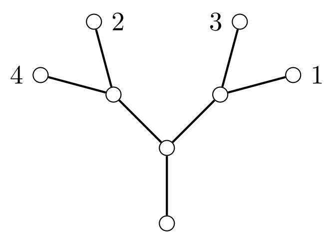

Figure 5: Not every map can be internally modified to obtain a homomorphism.

We can notice that no modification of only internal vertices can change h into a homomorphism to C_5.
In particular, this implies that there does not exist an operation which modifies vertex maps from $2T_k$ to C_5 only changing their values on internal vertices and has as output only graph homomorphisms.

4 Minor avoidance and homomorphisms to cycles

We know that the Pentagon Conjecture asks about the existence of homomorphisms between the family of cubic graphs and cycles. In this section, following Galluccio, Goddyn and Hell [16], we answer a similar question about the families of minor-avoiding graphs.
Besides the similarities with the Pentagon Conjecture, there is another perspective which we should point out. The study of the chromatic number of graphs is of the highest importance in Combinatorics (we can think of Brook's Theorem as an example); therefore, any generalisation or different point of view on the topic is of great interest on its own. In this section, we present the circular chromatic number, which is an extension to the rational numbers of the chromatic number of a graph. Crucially, we establish a link between low circular chromatic number and graph homomorphisms to cycles; we exploit this result studying the circular chromatic number of high-girth graphs avoiding a minor.

4.1 Circular chromatic number

Graph colouring is one of the main topics of interest in Combinatorics. In this subsection, we want to underline the link between graph colouring and graph homomorphisms, a link that holds also in the extension to circular colourings.
We observed in a previous section that any proper k-colouring c of a graph G naturally induces a homomorphism between G and K_k. Indeed, note that the map that associates to each vertex $v \in V(G)$ the vertex $c(v)$ in K_k is indeed a graph homomorphism (we assume $[k]$ to be the image set of c and the vertex set of K_k). The converse is also true, as we can regard the vertices of K_k as distinct colours, and observe that if c is a homomorphism between G and K_k, the same c is also a proper k-colouring.
This allows us to naturally generalise the concept of graph colouring. Indeed, we can ask ourselves for which graphs H a given graph G is H-colourable. We hope everyone agrees that this generalisation leads to interesting questions.
The main definition of this section is another extension of the concept of chromatic number; in particular, we define the circular chromatic number in a way that allows it to take values in the field of rational numbers. Moreover, the circular chromatic number is a strengthening of the definition of chromatic number in the sense that knowing the first allows us to calculate the latter but two graphs with the same chromatic number can have different circular chromatic numbers. This additional information contained in the circular chromatic number allows us to obtain some results about graphs homomorphisms.

Definition 4.1. Let C be the circle of length 1 around the origin of $\mathbb{R}^2$, or equivalently, $C = \partial B(0, \frac{1}{2\pi})$; and let $\phi : [0, 1) \to C$ be the natural map. For any positive real r, we denote with $C^{(r)}$ the set of open intervals of C of length $1/r$. Let G be a graph, we define an *r-circular colouring* of G as a map f from $V(G)$ to $C^{(r)}$ such that if x, y are adjacent vertices of G, then $f(x) \cap f(y) = \emptyset$.
The *circular chromatic number of* G, denoted by $\chi_c(G)$, is the infimum of the set of positive r for which G has an r-circular colouring (the minimum is attained, for a proof look at Zhu's survey [43]).

The circular chromatic number χ_c provides more information than the chromatic number χ; indeed, it holds $\lceil \chi_c(G) \rceil = \chi(G)$ (a proof of this fact relies on an alternative definition of circular chromatic number and can be found in [43, Theorem 1.1]) while it can be calculated that the flower snark J_5 and K_3 have the same chromatic number but distinct circular chromatic numbers.
The link between graph homomorphisms and graph colouring survives to the generalisation to circular colouring. In particular, because we are interested in graph homomorphisms to cycles, it is interesting to point out the following result.

Proposition 4.2 (Section 2, [43]). *Let G be a graph and let k be a positive integer. The following are equivalent:*

i) There exists a homomorphism $h : G \to C_{2k+1}$,

ii) There exists a $\left(2+\frac{1}{k}\right)$-circular colouring $c : G \to C^{(2+\frac{1}{k})}$.

Proof. We show that both conditions are equivalent to a third one, which is as follows.

iii) There exists a map $h' : V(G) \to \{0, \ldots, 2k\}$ such that if x, y are adjacent vertices of G it holds $|h'(x) - h'(y)|$ is either k or $k+1$.

For a general graph H, we denote with H^r the graph over the vertex set of H for which $xy \in E(H^r)$ if the distance between x and y in H is at least r. We can restate point iii) as: there exists a homomorphism between G and C_{2k+1}^k.
Because $2k+1$ is odd, it is not difficult to observe that C_{2k+1}^k is isomorphic as a graph to C_{2k+1}, and hence *i)* and *iii)* are equivalent.
For the equivalence between *ii)* and *iii)*, let us first show that *ii)* implies *iii)*. Let c be a $(2+\frac{1}{k})$-circular colouring for G. It simplifies the notation to study the induced map $c' : V(G) \to C$ (recall that $C = \partial B(0, \frac{1}{2\pi})$) which sends a generic vertex v to the central point of the interval $c(v)$ (which by definition is an interval of length $\frac{1}{2+1/k}$ in C).
The idea behind this result is that if we divide C into $2k+1$ segments of equal length, we work with them as distinct vertices of C_{2k+1} in the natural way, and we associate to $v \in V(G)$ the vertex corresponding to the segment in which $c'(v)$ lies, this association is indeed a graph homomorphism. More formally, we have as follows.
Recall that ϕ is the natural map between $[0, 1)$ and C. Let h be the map $h : V(G) \to \{0, \ldots, 2k\}$ defined as $h'(v) = \lfloor \phi^{-1}(c'(v)) \cdot (2k+1) \rfloor$. Let v, w be adjacent vertices in G; then the distance between $c'(v)$ and $c'(w)$ is at least $\frac{k}{2k+1}$ by definition of circular colouring. Therefore, the distance between $h'(v)$ and $h'(w)$ in the numbers modulo $2k+1$ at least k, as desired.
Let us now show that *iii)* implies *ii)*. Let h' be a map such as in *iii)*, and let c' be the map $c' : V(G) \to C$ defined as $c'(v) = \phi\left(\frac{h'(v)}{2k+1}\right)$. Because of the hypothesis in *iii)*, we have that adjacent vertices are sent by c' in points with "circular distance" at least $\frac{k}{2k+1}$. Hence, the natural corresponding map $c : V(G) \to C^{(2+1/k)}$, which sends v to a circular segment of length $\frac{k}{2k+1}$ centered in $c'(v)$, is a $\left(2+\frac{1}{k}\right)$-circular colouring of G. □

What we showed, is that graphs which are homomorphic to long cycles have circular chromatic number close to 2. Elaborating on this, suppose G is a graph

such that $\chi(G) = 3$, this means that $\chi_c(G) \in (2, 3]$, because $\lceil \chi_c(G) \rceil = \chi(G)$; in particular, this means that there exists k such that G is not homomorphic to C_{2k+1}. This should not surprise us, because a graph is bipartite if and only if it does not contain odd cycles.
Rewriting this remark from a different prospective, if we let $\mathcal{C}_{2k+1}$ be the class of graphs G such that $G \rightsquigarrow C_{2k+1}$. Then by the last proposition we can say that every graph in $\mathcal{C}_{2k+1}$ has circular chromatic number at most $2 + \frac{1}{k}$. In some sense, because we have $\mathcal{C}_3 \supset \mathcal{C}_5 \supset \dots$, we are tempted to say that the limit family of $\mathcal{C}_{2k+1}$ has chromatic number two. This is exactly what happens. Indeed, because every bipartite graph has a homomorphism to C_{2k+1} for any k, and because every graph containing an odd cycle (every non-bipartite graph) is not homomorphic to C_{2k+1} for k large enough, we have

$$\bigcap_{k \in \mathbb{N}^+} \mathcal{C}_{2k+1} = \{G \colon \chi(G) \leq 2\} .$$

4.2 Girth-bipartite families and p-path degenerate graphs

For what we said in this last remark, it is reasonable to give the following definition.

Definition 4.3 (Girth-bipartite)**.** Let $\mathcal{G}$ be a class of graphs, we say that $\mathcal{G}$ is *girth-bipartite* (or almost bipartite) if for any $\varepsilon > 0$ there exists $g_\varepsilon \in \mathbb{N}$ such that every graph in $\mathcal{G}$ of girth at least g_ε has circular chromatic number at most $2 + \varepsilon$.

In Section 2, we proved that the class of cubic graphs is not almost bipartite. The main result of this section, due to Galluccio et al., states that almost bipartite families are not so rare.

Theorem 4.4 (Galluccio et al. [16])**.** *For any fixed graph H, the class of H-minor free graphs is girth-bipartite.*

We should take a moment to point out the relevance of this result. Indeed, the study families of H-minor free graphs is quite interesting by itself. A famous result by Kuratowski states that any K_5 and $K_{3,3}$-minor avoiding graph is planar and vice versa. Moreover, these families are also studied in relation to separators because of their sparsity (see the famous Planar Separator Theorem by Lipton and Tarjan [24] and its generalisation by Alon et al. [2]).
It is interesting to point out that this last theorem implies that the family of planar graphs is almost bipartite (this, combined with the result of Section 2, and the equivalence we proved in the last subsection, gives us that a cubic graph is almost surely not planar); more generally, every infinite subfamily of a minor-avoiding family is almost bipartite.
Before proceeding to the proof of the theorem, we need the definition of p-path degenerate graph. We then prove that path degeneracy is a sufficient condition for a graph to have a homomorphism to some cycle and that graphs of high girth avoiding minors are indeed path degenerate.

Definition 4.5. Let G be a graph and p a positive integer. G is said to be *p-path degenerate* if there exists a finite sequence $G = G_0, \dots, G_t$ of 2-connected subgraphs of G such that G_t is bipartite and each G_i is constructed from G_{i-1}

in the following way. There is a path $P = u_1, \dots, u_\ell$ in G_{i-1} with $\ell \geq p$, for which all internal vertices $U = \{u_2, \dots, u_{\ell-1}\}$ have degree exactly two, and we have $G_i = G_{i-1} \setminus U$.

The following lemma allows us to use path degeneracy to study the circular chromatic number.

Lemma 4.6. *Let G be a graph, ℓ an odd positive integer, $P = u_1, \dots, u_p$ a path in G with $p \geq \ell$ such that every internal vertex of P has degree exactly two and U the set of internal vertices of P. Then $G \rightsquigarrow C_\ell$ if and only if $G \setminus U \rightsquigarrow C_\ell$.*

Proof. For any homomorphism of G to C_ℓ, its restriction to a subgraph of G is still a homomorphism. Therefore if $G \rightsquigarrow C_\ell$, then it also holds $G \setminus U \rightsquigarrow C_\ell$.
On the other hand, let $h : V(G \setminus U) \to V(C_\ell)$ be a homomorphism as in the hypothesis. Because ℓ is odd, $h(u_1)$ and $h(u_p)$ divide C_ℓ in two paths (may be trivially), one of odd and one of even length. Let $h(u_1) = v_1, \dots, v_q = h(u_p)$ be the path Q of C_ℓ of the same parity of p; then, because $q \leq \ell \leq p$, we have that there exists a homomorphism between U and Q which extends h. This naturally induces a homomorphism from G to C_ℓ because all the vertices of U do not have neighbours outside P. □

The next corollary follows from the last lemma, and it of easier application.

Corollary 4.7. *Let G be a p-path degenerate graph, then $G \rightsquigarrow C_\ell$ with ℓ odd of size at most $p + 1$. In particular, by Proposition 4.2 we have*

$$\chi_c(G) \leq 2 + \frac{1}{\lfloor p/2 \rfloor}.$$

Proof. Let $G = G_0, \dots, G_t$ be the sequence given to us by the definition of p-path degeneracy and ℓ be the largest odd integer smaller than $p + 1$. Because B_t is bipartite, we have $B_t \rightsquigarrow C_\ell$. Moreover, by construction of G_i from G_{i-1} and by Lemma 4.6 we have that $G_0 \rightsquigarrow C_\ell$ if and only if $G_t \rightsquigarrow C_\ell$. This allows us to conclude that $G_0 \rightsquigarrow G_\ell$.
The bound on the circular chromatic number follows from Proposition 4.2. □

4.3 Minor-avoiding families are almost bipartite

We prove Theorem 4.4 using the tools introduced in the last subsection, in particular Corollary 4.7.
The general idea behind the proof of Theorem 4.4 is to prove that any graph of high enough girth in a minor-avoiding family is p-path degenerate for some p depending only on the girth bound. We need a technical lemma (for a proof, see Thomassen [39, pp. 115]) that allows us to find some vertices of degree two in said graphs (remember that we want to find paths in which all the internal vertices have degree two).

Lemma 4.8 (Thomassen, [39])**.** *Let H be a graph. There exists a positive integer k such that every H-minor free graph G with $\delta(G) \geq 3$ has girth at most k.*

Let us denote with k_H be the minimum of such k; we have that any H-minor free graph with girth at least $k_H + 1$ has $\delta(G) \leq 2$.

Lemma 4.9. *Let G be an H-minor free graph of girth at least $k_H(p-1)+1$ for p a positive integer. Then G is p-path degenerate.*

Proof. We may assume without loss of generality that G is a 2-connected graph because a graph is not p-path degenerate only if it has a 2-connected component that is not p-path degenerate.
Let $Q(G)$ be the set of vertices of G of degree at least three. We prove this lemma by induction on $f(G) = \left(\sum_{v \in Q(G)} d(v)\right) - 2\,|Q(G)|$. The base cases are the ones in which G is a cycle, and the statement trivially holds in these.
Suppose G is not a cycle. Let $G_0 = G$ and let $x, y, z \in V(G_0)$ such that $N_{G_0}(y) = \{x, z\}$; we define G_1 over $V(G_0) \setminus \{y\}$ by contracting the vertex y to the vertex x (we add the edge xz and remove the vertex y). We construct G_i from G_{i-1} in the same way if there is a vertex of degree 3 in G_{i-1}. Let G' be the last graph we obtain; observe that $\delta(G') \geq 3$.
Because G is H-minor free, and because G' is a minor of G, we also have that G' is H-minor free. Therefore, we can apply Lemma 4.8 to state that there exists a cycle C' in G' of length at most k_H. This cycle corresponds in G in a natural way to a cycle C which has length at least $k_H(p-1)+1$ (which is the girth of G). Because in C there are only at most k_H vertices of degree three or more, by pigeonhole theorem this means that G has a path P of length at least p with internal vertices U of degree exactly two.
We can apply the inductive hypothesis on $G \backslash U$, which is an H-minor free graph with girth at least $k_H(p-1)+1$ and $f(G \backslash U) < f(G)$. This allows us to conclude by Lemma 4.6. □

Proof of Thm 4.4. Follows from Lemma 4.9 and Corollary 4.7. Indeed, if ℓ a positive integer such that $\frac{1}{\ell} < \varepsilon$, we can take the family $\mathcal{G}$ of H-minor free graphs of girth at least $k_H(2\ell-1)+1$. Every graph in this family is 2ℓ-path degenerate by Lemma 4.9, and hence by Corollary 4.7 we have:

$$\chi_c(G) \leq 2 + \frac{1}{\ell} < 2 + \varepsilon.$$

□

5 A variation of the Pentagon Problem

In this section, we present our work about the Pentagon Conjecture. In the first part, we give a generalisation of the Pentagon Problem that allows for approximated results; we then use a local approach (inspired by DeVos and Šámal [10]) to study an upper bound on the error. In the last part, we proceed to underline the problems with this approach and to explain why it cannot be immediately used to solve the Pentagon Conjecture.

5.1 Approximating the Pentagon Conjecture

Having the goal of approximating the Pentagon Problem, we need a tool to measure how much a map fails to be a homomorphism. Ideally, for any cubic graph G we would like to have the map $h : V(G) \to V(C_5)$ which violates the minimum number of edges.

Definition 5.1. Let G and H be graphs, and let $h : V(G) \to V(H)$. We denote with S_h the set $\{uv \in E(G) : h(u)h(v) \notin E(H)\}$ of violated edges. We have that h is an homomorphism if and only if $S_h = \emptyset$. Because studying S_h is as difficult as studying h, we use the proportion $\frac{S_h}{|E(G)|}$ as measure of how much h fails to be an homomorphism. We denote $\omega_h = \frac{|S_h|}{|E(G)|}$.

The Pentagon conjecture can be then formulated as follows. If G is a cubic graph of girth high enough, there exists a map $h : V(G) \to V(C_5)$ such that $|S_h| = 0$ (or, equivalently $\omega_h = 0$). This suggests us some other definition.

Definition 5.2. Let G be a graph; we denote with $\omega_*(G)$ the minimum of the set $\{\omega_h \text{ s.t. } h : V(G) \to V(C_5)\}$.

With this notation, the Pentagon Problem can be formulated as follows. Is it true that $\omega_*(G) = 0$ for any G which is cubic and with girth high enough?
We examine the following weakening of the Pentagon Conjecture.

Conjecture 5.3. *For every ε positive real, there exists $N \in \mathbb{N}$ such that if G is a cubic graph with girth$(G) \geq N$, then $\omega_*(G) \leq \varepsilon$.*

If the Pentagon Conjecture holds, then so does this weakening; while the viceversa is not true. Let G be a triangle-free cubic graph, $h : V(G) \to V(C_5)$ any map and $v \in V(G)$. It is interesting to notice that by changing the value of $h(v)$ we can ensure that h violates at most one of the edges adjacent to v. It follows that $\frac{1}{3}$ is an upper bound for $\omega_*(G)$ whenever G is a triangle-free cubic graph. In this section, we present a local approach to the previous conjecture, and we find a better bound for $\omega_*(G)$. However, the upper bound we show is bounded away from 0. Indeed, we prove the following.

Proposition 5.4. *For evey $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that if G is a cubic graph of girth at least $2k + 1$, then $\omega_*(G) \leq \frac{1}{4}(1 + \varepsilon)$.*

5.2 Local approach

We start with some terminology. Firstly, we use here the notation about cubic trees introduced in Section 3; moreover, we frequently refer to levels of T_k, both

for edges and for vertices. In particular, we denote with level 0 the root vertex and the root edge; we denote level 1 the vertex adjacent to the root vertex and the two edges adjacent to it which are not in level 0. Generally, we define level $i+1$ as the set of vertices not in level $i-1$ which are adjacent to the vertices in the level i; for edges, the level $i+1$ is defined as the set of edges not in level i adjacent to the vertices in the $i+1$-th level.
The main idea behind our approach is as follows. Let G be a cubic graph with high enough girth, and let $h : V(G) \to V(C_5)$ such that $\omega_h = \omega_*(G)$. It holds that if $T \subseteq V(G)$ is a set of vertices spanning a connected subtree of $V(G)$, then there is no homomorphism $h' : T \to V(C_5)$ such that $|S_{h'}| < |S_h \cap E(T)|$ and h and h' coincide on the leaves of T (or, as we often write, $h\big|_{\partial T} = h'\big|_{\partial T}$); indeed, if this were not the case, the map $\overline{h} = h\big|_{V(G)\setminus T} \cup h'\big|_{T}$ would be such that $\omega_{\overline{h}} < \omega_h = \omega_*(G)$.
We show that if ω_h is bigger than a certain value, we can find a subtree T and a map h' as in the last paragraph.

Definition 5.5. Let G be a graph. A *weight* of the edges of T is a map $\mu : E(G) \to \mathbb{R}^+$. For $S \subseteq E(G)$ we write $\mu(S)$ to indicate $\sum_{e \in S} \mu(e)$. We always assume that all the edges have strictly positive weight and that $\mu(E(G)) = 1$. Given a generic map $\nu : E(G) \to \mathbb{R}^+$, we write $\mu \simeq \nu$ to indicate that μ is the normalised weight $\mu : e \mapsto \frac{\nu(e)}{\nu(E(G))}$.

We are now ready for the first step of our approach. The following proposition, given a map $V(G) \to V(C_5)$ with many violated edges, provides the existence of a subtree T of G with many violated edges; this allows us to work locally. Before presenting the result, it is useful to agree on some notation.

Notation. Let $H = (V, E), H' = (V', E')$ be two graphs and let $\mu : E \to \mathbb{R}, \mu' : E' \to \mathbb{R}$ be two edge weights. We say that H and H' are weight isomorphic if there exists an isomorphism $g : V \to V'$ which preserves the weight of the edges.

We present a general result.

Proposition 5.6. *Let G be a graph, let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a set of edge-weighted subtrees of G which are pairwise weight isomorphic, and for every $e \in E(G)$ let $\mu^*(e) = \sum_{H \in \mathcal{H}} \mu_H(e)$. Moreover, suppose that $\mu^*(e)$ does not depend on e. If $S \subseteq E(G)$ is such that $|S| \geq \alpha\,|E(G)|$, then there exists $H' \in \mathcal{H}$ such that $\mu_{H'}(S \cap E(H')) \geq \alpha$.*

Proof. Observe that it holds:

$$\begin{aligned}\sum_{H\in\mathcal{H}} \mu_H(S\cap E(H)) &= \sum_{e\in S}\sum_{H\in\mathcal{H}} \mu_H(e)\\ &= \sum_{e\in S}\mu^*(e) = \mu^*(e)\,|S|\\ &\geq \mu^*(e)\alpha\,|E(G)|\\ &= \alpha \sum_{e\in E(G)}\sum_{H\in\mathcal{H}} \mu_H(e)\\ &= \alpha \sum_{H\in\mathcal{H}}\sum_{e\in E(G)} \mu_H(e)\\ &= \alpha.\end{aligned}$$

Therefore, it cannot be the case that $\mu_{H'}(S\cap E(H')) < \alpha$ for every $H' \in \mathcal{H}$. □

This proposition can be applied to our case.

Corollary 5.7. *Let G be a cubic graph of girth at least $2k+1$, and let μ be a weight for T_k homogeneous on levels (for any fixed level, each edge in that level has the same weight). If $h : V(G) \to V(C_5)$ violates $|S_h| = \alpha\,|E(G)|$ edges, then there exists a subtree H of G isomorphic to T_k such that $\mu(S_h \cap H) \geq \alpha$.*

Proof. Let $\mathcal{H}$ be the set of all subgraphs of G isomorphic to T_k; moreover, for any edge e, let $N^{(i)}(e)$ be the graph induced by G over the vertices with distance at most k from any one of the vertices adjacent to e. Because the girth of G is at least $2k+1$, we have that all the $N^{(k)}(e)$ are isomorphic to $2T_k$. Therefore, in particular, $\mu^*(f) = \sum_{H\in\mathcal{H}}\mu(f)$ is constant for every edge f.
Therefore, by our last lemma, we can find a copy H of T_k such that $\mu(S_h\cap H) \geq \alpha$ as required. □

5.3 A first application

Let G be a cubic graph of girth at least $2k+1$; let $\nu : E(T_k) \to \mathbb{R}^+$ the constant function $\nu : e \mapsto 1$ and let $\mu_o \simeq \nu$ the normalised homogeneous weight.
We can apply Corollary 5.7 to this situation; we first need to formally introduce an argument that we mentioned above.

Remark 5.8. Let G and T_k be as above. If $h : V(G) \to V(C_5)$ is a map such that $\omega_h = \omega_*(G)$, then there is no subgraph T of G isomorphic to T_k such that there exists $h' : V(T) \to V(C_5)$ such that $|S_{h'}| < |S_h \cap T|$ and $h\big|_{\partial T} = h'\big|_{\partial T}$.
We can prove this by considering the vertex map $\overline{h} = h\big|_{V(G)\setminus T} \cup h'\big|_{T}$. Indeed, if such a subgraph T existed, we would have $\omega_{\overline{h}} < \omega_*(G)$.

The main idea behind the result of this subsection is as follows.

Lemma 5.9. *Let $h : V(T_k) \to V(C_5)$ such that $\omega_h > \frac{2^{k-2}}{2^k-1}$. Then there exists $h' : V(T_k) \to V(C_5)$ such that $h\big|_{\partial T_k} = h'\big|_{\partial T_k}$ and $|S_{h'}| < |S_h|$.*

Proof. Without loss of generality, by vertex transitivity of C_5, we can assume that $h(\text{root}) = 1$. Let h' be the partial colouring of T_k defined only on the vertex levels $0, 1, 2\ldots, k-1, k+1$ which takes alternatively values 2 and 1 in each level.

More specifically, we impose $h\big|_{\partial T_k} = h'\big|_{\partial T_k}$, then, for the levels $1, \dots, k-1$ we define $h'(v) = 1$ if v is on an even level, and $h'(v) = 2$ if v is on an odd level. As the example in Figure 6.

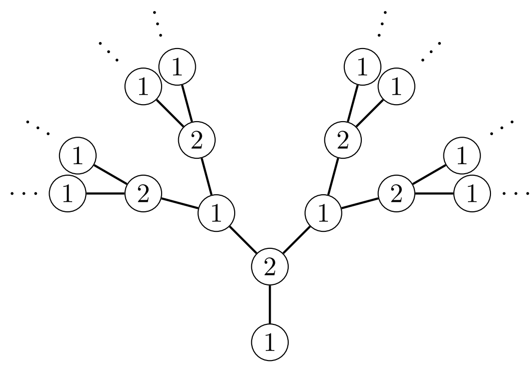

Figure 6: An example-map.

However we complete h', it clearly holds that $S_{h'}$ is a subset of the two last edge levels. In order to show that we can complete h' in such a way that $\omega_{h'} \leq \frac{2^{k-2}}{2^k-1}$ it suffices to show that for any value on the leaves of T_2, it is always possible to assign to the middle vertex a value such that at most one edge is violated in T_2. The reason because this suffices is that to complete h' we just need to colour each vertex in the k-th level. Each of these vertices is the central vertex in a different copy of T_2, and these copies are edge disjoint. If we prove the aforementioned claim for T_2, we can then complete h' in such a way that there is at most one violated edge for each vertex of the k -level. To show that this is indeed true, first observe that T_2 has three leaves and one middle vertex. If two of the leaves are labelled with the same vertex of C_5, then it is clearly possible to assign to the middle vertex of T_2 a vertex of C_5 such that at most one edge of T_2 is violated. On the other hand, if all the leaves of T_2 are associated to distinct vertices of C_5, by pidgeonhole theorem at least two of them are in the form $i, i+2$ and therefore it is still possible to complete the map in such a way that at most one vertex is violated.
From this last consideration, we have that we can complete h' in such a way that at most 2^{k-2} edges are violated. $\square$

We are now ready to prove our upper bound on $\omega_*(G)$.

Proof of Proposition 5.4. Let k be such that $\frac{2^k}{2^k-1} < (1+\varepsilon)$. Suppose by contradiction that $\omega_*(G) > \frac{1}{4}(1+\varepsilon)$; this means that for every $h : V(G) \to V(C_5)$, the map h violates at least 2^{k-2} edges. Let μ_o be the weight on the edges of T_k and let h be such that $\omega_h = \omega_*(G)$; then by Corollary 5.7 we have that there exists $T \subseteq G$ a subtree of G such that $\mu(T \cap S_h) \geq \frac{1}{4}(1+\varepsilon)$.
Recalling the definition of μ_o, this means that $|S_h \cap T| > 2^{k-2}$. By Lemma 5.9 we have that there exists a map $h' : V(T) \to V(C_5)$ which is equal to h on ∂T and with strictly less violated vertices. Therefore we can extend h' to $V(G)$ and obtain a contradiction by minimality of ω_h. $\square$

5.4 Possible generalisations

The idea behind the localised approach to Conjecture 5.3 is quite straightforward. We call a map $h : V(T_k) \to V(C_5)$ a bad map if it does not exists $h' : V(T_k) \to V(C_5)$ such that $|S_{h'}| < |S_h|$ and $h\big|_{\partial T_k} = h\big|_{\partial T_k}$. For a level-homogeneous weight μ of the edges of T_k, we use the following notation:

$$\omega^*(\mu) = \max\{\mu(S_h) : h \text{ is bad}\}.$$

Moreover, we denote:

$$\Omega(k) = \min\{\omega^*(\mu) : \mu \text{ is a level-homog. edge weight of } T_k\}.$$

The local approach that we presented is equivalent to the following remark.

Remark 5.10. If G is a cubic graph of girth at least $2k+1$, then $\omega_*(G) \le \Omega(k)$.

Proof. Let $h : V(G) \to V(C_5)$ such that $\omega_h = \omega_*(G)$; and let μ such that $\omega^*(\mu) = \Omega(k)$.
Suppose by sake of contradiction that $\omega_*(G) > \Omega(k)$. Then, by Proposition 5.6 and by a consideration already seen in the proof of Corollary 5.7, we have that there exists $T \subseteq G$ isomorphic to T_k such that $\mu(T \cap S_h) \ge \omega_h = \omega_*(G) > \Omega(k)$. There are two possibilities:

- Either $h\big|_T$ is bad, in which case we would have $\mu(T \cap S_h) \le \omega^*(\mu) = \Omega(k)$, which is a contradiction;
- or $h\big|_T$ is not bad. But in this case we can find $h' : V(G) \to V(C_5)$ such that $\omega_{h'} < \omega_h$, which again leads to contradiction.

□

What we did in the previous subsection can be summarised by saying that we showed that $\omega^*(\mu_o) = \frac{2^{k-2}}{2^k-1}$, and therefore that $\Omega(k) \le \frac{2^{k-2}}{2^k-1}$.
The local approach fails to provide an immediate proof to Conjecture 5.3; indeed, it is not easy to determine the set of bad maps $h : V(T_k) \to V(C_5)$. We know, for example, that if h is bad then:

i) $|S_h| \le 2^{k-2}$,

ii) every vertex in T_k is adjacent to at most one edge in S_h,

iii) if T is subtree of T_k isomorphic to T_i, then $|S_h \cap T| \le 2^{i-2}$.

But, on the other hand, given a map h with the properties just presented, it is not immediate to determine whether h is bad or not. Moreover, by analysing an inductive definition of an S_h with the three aforementioned characteristics it seems reasonable that for every μ we have

$$\max\{\mu(S_h) : h \text{ satisfies } i),\ ii),\ iii)\} \ge \frac{2^{k-2}}{2^k-1} = \omega^*(\mu_o).$$

For this reason we believe that this local approach does not provide any stronger result.

6 Chromatic number of triangle-free graphs

While working on the Pentagon Problem, we encountered the problem of studying the chromatic number of triangle-free regular graphs. In this section, we analyse a famous result about a similar problem.
For a graph G of girth at least 5, let G^3 be the graph over $V(G)$ for which $xy \in E(G^3)$ if and only if the distance of x and y in G is exactly 3. Observe that if G is a cubic graph and if $f : V(G) \to V(C_5)$ is a homomorphism, then there is no pair of vertices x, y at distance exactly 3 such that $f(x) = f(y)$; in particular, $\chi(G^3) \leq 5$. Inspired by this observation (and by the fact that G^3 is 12-regular and without triangles if G is cubic of high enough girth), in this section we present a result about the colourability of triangle-free graphs.

6.1 Presentation of the result

Recently, Molloy [30] presented a refined version of an unpublished result by Johansson [19] about the chromatic number of triangle-free graphs with high maximum degree. With his new approach, the author uses entropy compression to obtain an easier and shorter proof.
We prove a statement weaker than the one in [30], by following the argument presented by Bernshteyn in [4]; in this last article, the author shows a more immediate approach to the problem of Johansson and Molloy, and also gives a generalised statement (that we do not analyse here).
The main result of this section is as follows.

Theorem 6.1 (Johannson [19], Molloy [30]). *For every $\varepsilon \in \mathbb{R}^{>0}$, there exists $\Delta_\varepsilon \in \mathbb{N}$ such that every triangle-free graph G of maximum degree $\Delta \geq \Delta_\varepsilon$ has the following bound on its chromatic number:*

$$\chi(G) \leq (1+\varepsilon)\frac{\Delta}{\ln \Delta}.$$

The articles by Molloy [30] and Johannson [19] directly treat the more general case of list chromatic number, obtaining the same upper bound; while in his article, Bernshteyn [4] extends this result for DP-colourings. What we study in this section is a strict weakening of the results presented in the mentioned articles, but it is still interesting to analyse.
In the first subsection, we present Lovász Local Lemma, one of its alternative presentations called Lopsided Lovász Local Lemma due to Erdős and Spencer [15], and a modification for negatively correlated variables of the well-known Chernoff bounds. In the second subsection, we present the general idea behind the proof of Theorem 6.1 due to Bernshteyn. In the last subsection, we present a proof of Lemma 6.12, from which the theorem follows naturally.

6.2 Lopsided Lovász Local Lemma and Chernoff Bounds

In this subsection, we present two variations of well-known theorems. We provide a proof of the first of these following the line of [15], we referred to Panconesi and Srinivasan [36] for the proof of the second one. We also present the general form for the first of the two results (though without proving it), because of its wide applicability in Combinatorics.

6.2.1 General Lovász Local Lemma and Lopsided version

To understand the idea behind the Lovász Local Lemma, we propose the following example.

Example 6.2. Let $B_1, B_2, \dots$ be a sequence of mutually independent events in a probability space, and suppose that each B_i has probability at least p for some fixed positive real p. Moreover, let $A_n = \bigcap_{i \le n} B_i$.
By definition of independence, for every $n \in \mathbb{N}$ the probability of A_n is positive (it is at least p^n), even if it can be exponentially small in n. One interesting consequence is that the intersection of a finite number of B_i is never empty.

Lovász Local Lemma is a powerful lemma that provides a similar result in a weaker, finite, setting. This kind of results is particularly important in Combinatorics because we often want to prove that an event occurs with positive probability.

Definition 6.3. Let $A_1, \dots, A_n$ be events in a generic probability space, and let $D = ([n], E)$ be a directed graph: moreover, let us take for every $i \in [n]$, a subset $S_i = \{j \in [n] \setminus \{i\} : ij \notin E\}$. We say that D is a dependency digraph for $A_1, \dots, A_n$ if each A_i is mutually independent of the set $\{A_j\}_{j \in S_i}$. More specifically, if for any subset $S \subseteq S_i$ it holds:

$$\mathbb{P}\left[A_i \cap \bigcap_{j \in S} A_j\right] = \mathbb{P}[A_i]\mathbb{P}\left[\bigcap_{j \in S} A_j\right].$$

We are now ready to state Lovász Local Lemma (for a reference, [3, Lemma 5.1.1]).

Lemma 6.4 (Lovász Local Lemma)**.** *Let $A_1, \dots, A_n$ be events in some probability space, and let $D = ([n], E)$ be a dependency digraph for them. Suppose there exists $x \in [0,1)^n$ such that $\mathbb{P}[A_i] \le x_i \prod_{ij \in E}(1 - x_j)$ for each $i \in [n]$; then it holds:*

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{A}_i\right] \ge \prod_{i=1}^{n}(1 - x_i) > 0.$$

The main observation is that, in order to obtain $\bigcap_i \overline{A}_i \neq \emptyset$, we do not require for all the events to be mutually independent, we just need that the dependencies are rare enough; indeed it is sometimes the case that we can prove that events which are in some sense “far” from each other are independent.
In the lopsided version of the Local Lemma, every mention of the independence is removed.

Definition 6.5. Let $A_1, \dots, A_n$ be events in a probability space. Suppose $\Gamma = ([n], E)$ is a simple graph such that for any $i \in [n]$ and for any $S \subseteq [n] \setminus (\{i\} \cup N_\Gamma(i))$, we have:

$$\mathbb{P}\left[A_i \cap \bigcap_{j \in S} \overline{A}_j\right] \le \mathbb{P}[A_i]\mathbb{P}\left[\bigcap_{j \in S} \overline{A}_j\right].$$

Then we say that Γ is a lopsidependency graph for $A_1, \dots, A_n$.

We are finally ready for the statement of the Lopsided Lovász Local Lemma.

Lemma 6.6 (LLLL, [15])**.** *Let $A_1, \ldots, A_n$ be nontrivial events in a probability space, and let Γ be a lopsidependency graph for them with maximum degree d. Suppose there exists $p \in [0,1]$ such that $\mathbb{P}[A_i] \leq p$ for every $i \in [n]$; and suppose also that $4dp \leq 1$. Then the event $\bigcap_{i=1}^{n} \overline{A}_i$ has positive probability.*

Proof. We proceed by induction on n, the base case being trivial. By definition of conditional probability,

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{A}_i\right] = \prod_{i=1}^{n} \mathbb{P}\left[\overline{A}_i \middle| \bigcap_{j=1}^{i-1} \overline{A}_j\right].$$

We notice that in the above equation, $\bigcap_{j=1}^{i-1} \overline{A}_j$ is nontrivial because of our inductive hypothesis.
We show that for any $i \in [n]$ and for any $S \subseteq [n]$, it holds $\mathbb{P}\left[A_i \middle| \bigcap_{j \in S} \overline{A}_j\right] \leq 2p$.
This result, used in the above equation, provides,

$$\begin{aligned}\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{A}_i\right] &= \prod_{i=1}^{n} \mathbb{P}\left[\overline{A}_i \middle| \bigcap_{j=1}^{i-1} \overline{A}_j\right] \\ &\geq \prod_{i=1}^{n} (1-2p) > 0.\end{aligned}$$

We show by strong induction on s that for any $i \in [n]$ and for any $S \subseteq [n]$ of size s, we have $\mathbb{P}\left[A_i \middle| \bigcap_{j \in S} \overline{A}_j\right] \leq 2p$.
The cases in which $s = 0$ or $i \in S$ are trivial. Hence, we may assume $s > 0$ and $i \notin S$. Therefore, up to rearrangement of the indices of the events, we may assume the following: $i = n$, also $S = \{1, \ldots, s\}$ and $N_\Gamma(i) \cap S = \{1, \ldots, d_*\}$ with $d_* \leq d$. With this in mind, and again using the definition of conditional probability, we can write

$$\mathbb{P}\left[A_n \middle| \bigcap_{j=1}^{s} \overline{A}_j\right] = \frac{\mathbb{P}\left[A_n \wedge \bigcap_{j=1}^{d_*} \overline{A}_j \middle| \bigcap_{h=d_*+1}^{s} \overline{A}_h\right]}{\mathbb{P}\left[\bigcap_{j=1}^{d_*} \overline{A}_j \middle| \bigcap_{h=d_*+1}^{s} \overline{A}_h\right]}.$$

As above, the events for which we condition are nontrivial by inductive hypothesis. We provide an upper bound for the numerator and a lower bound for the denominator as follows.

- For the numerator, it holds,

  $$\mathbb{P}\left[A_n \wedge \bigcap_{j=1}^{d_*} \overline{A}_j \middle| \bigcap_{h=d_*+1}^{s} \overline{A}_h\right] \leq \mathbb{P}\left[A_n \middle| \bigcap_{h=d_*+1}^{s} \overline{A}_h\right] \leq \mathbb{P}[A_n] \leq p.$$

  The first inequality holds by monotonicity of probability measures; the second follows by definition of lopsidependency graph and because in our rearrangement of indices we have $N_\Gamma(i) \cap S = \{1, \ldots, d_*\}$; while the last inequality holds by hypothesis.

- We can proceed as follows about the denominator:

$$\begin{aligned}\mathbb{P}\left[\bigcap_{j=1}^{d_*}\overline{A}_j\middle|\bigcap_{h=d_*+1}^{s}\overline{A}_h\right] &\geq 1-\sum_{j=1}^{d_*}\mathbb{P}\left[A_j\middle|\bigcap_{h=d_*+1}^{s}\overline{A}_h\right]\\ &\geq 1-\sum_{j=1}^{d_*}2p \geq 1-2pd\\ &\geq \frac{1}{2}.\end{aligned}$$

Indeed, the first inequality follows because, for general A, B events in a probability space, $\mathbb{P}[A\cap B]+\mathbb{P}[\overline{A}\cup\overline{B}] = 1$. The second inequality follows by strong induction hypothesis on s of the claim we are proving; while the last two inequalities are by hypotesis.

Therefore, by these observations we obtain

$$\mathbb{P}\left[A_n\middle|\bigcap_{j=1}^{s}\overline{A}_j\right] \leq 2p.$$

□

Before we introduce the next subsection, it is useful to point out a general remark, that might enlighten us on how we can apply Lemma 6.6.

Remark 6.7 (Law of total probability). Let $B_1, \dots, B_n$ be a nontrivial partition of a probability space. Then, for any event A on the same space, we can write:

$$\mathbb{P}[A] = \sum_{i=1}^{n}\mathbb{P}\left[A|B_i\right]\mathbb{P}[B_i].$$

Therefore, if we want to prove $\mathbb{P}[A] \leq p$ for some event A and some $p \in [0,1]$, it suffices to show that for every $i \in [n]$, it holds $\mathbb{P}\left[A|B_i\right] \leq p$.
This law of total probability also holds for conditional probabilities. More specifically, suppose C is a nontrivial event, in the same probability space, we have that

$$\mathbb{P}\left[A|C\right] = \sum_{C\cap B_i\neq\emptyset}\mathbb{P}\left[A|C\cap B_i\right]\mathbb{P}\left[B_i|C\right].$$

Since $\sum_{C\cap B_i\neq\emptyset}\mathbb{P}\left[B_i|C\right] = 1$, if we want to show that $\mathbb{P}\left[A|C\right] \leq p$ is suffices to show that $\mathbb{P}\left[A|C\cap B_i\right] \leq p$ for every B_i such that $B_i \cap C \neq \emptyset$.
This last version of the law of total probability becomes particularly useful when we have an event C and a partition $B_1, \dots, B_n$ such that for every $i \in [n]$ it holds either $B_i \subseteq C$ or $B_i \cap C = \emptyset$. Indeed, in this case in order to show that $\mathbb{P}\left[A|C\right] \leq p$ it suffices to show that $\mathbb{P}\left[A|B_i\right] \leq p$ for every B_i such that $B_i \subseteq C$.

We present another, quite technical, Remark which we use in what follows.

Remark 6.8. Let A, B be nontrivial events in some probability space; $\mathbb{P}\left[B|\overline{A}\right] \geq \mathbb{P}[B]$ if and only if $\mathbb{P}\left[A|B\right] \leq \mathbb{P}[A]$.

Proof. The two conditions are equivalent; indeed we can write:

$$\begin{aligned}
\mathbb{P}\left[A|B\right] \le \mathbb{P}[A] \qquad & \text{iff} \\
\mathbb{P}\left[A \cap B\right] \le \mathbb{P}[A]\mathbb{P}[B] \qquad & \text{iff} \\
\mathbb{P}\left[\overline{A} \cap B\right] \ge \mathbb{P}[B]\mathbb{P}[\overline{A}] \qquad & \text{iff} \\
\mathbb{P}\left[B \cap \overline{A}\right] \ge \mathbb{P}[B]. &
\end{aligned}$$

□

We are now ready for Chernoff bounds.

6.2.2 Chernoff bounds

Following an article of Panconesi and Srinivasan [36], we now present a modified version of the well-known Chernoff bounds as shown in [30].

Definition 6.9. Let $X_1, \dots, X_n$ be $\{0,1\}$-valued random variables. We say that they are negatively correlated if, for any nonempty set $S \subseteq [n]$, we have:

$$\mathbb{P}[\forall i \in S,\ X_i = 1] \le \prod_{i \in S} \mathbb{P}[X_i = 1].$$

Or, equivalently, if for any such S it holds $\mathbb{E}\left[\prod_{i\in S} X_i\right] \le \prod_{i\in S} \mathbb{E}[X_i]$.

We present now a technical detail that we need in the proof of next lemma; for a reference, see Lee [23, Lecture 5]

Claim. Let $Y_1, \dots, Y_n$ be negatively correlated $\{0,1\}$-valued random variables, and let $Y = \sum_{i=1}^n Y_i$. For any $h > 0$, it holds

$$\mathbb{E}\left[e^{hY}\right] \le \prod_{i=1}^n \mathbb{E}\left[e^{hY_i}\right].$$

Proof. By linearity of expectation and by the close formula for powers of sums we can write, for $k \in \mathbb{N}^+$:

$$\mathbb{E}\left[Y^k\right] = \sum_{\alpha \in (\mathbb{N})^n : \|\alpha\|_1 = k} \mathbb{E}\left[Y_1^{\alpha_1} \cdot \ldots \cdot Y_n^{\alpha_n}\right] \le \sum_{\alpha \in (\mathbb{N})^n : \|\alpha\|_1 = k} \prod_{i=1}^n \mathbb{E}\left[Y_i^{\alpha_i}\right].$$

The inequality follows because the variables are negatively correlated and have values in $\{0,1\}$.
Let $\overline{Y}_1, \dots, \overline{Y}_n$ be independent random variables such that for every i the image distribution of $\overline{Y}_i$ equals the image distribution of Y_i. We can rewrite the above centred equation as $\mathbb{E}\left[Y^k\right] \le \mathbb{E}\left[\overline{Y}^k\right]$; indeed, because the $\overline{Y}_i$ are independent, we have that the right hand side of the above equation is equal to $\mathbb{E}\left[\overline{Y}^k\right]$. Then

we have, by Taylor's theorem:

$$\begin{aligned}\mathbb{E}\left[e^{hY}\right] &= \sum_{i=0}^{\infty} \frac{h^i \mathbb{E}[Y^i]}{i!} \leq \sum_{i=0}^{\infty} \frac{h^i \mathbb{E}[\overline{Y}^i]}{i!} \\ &= \mathbb{E}\left[\prod_{i=1}^{n} e^{h\overline{Y_i}}\right] = \prod_{i=1}^{n} \mathbb{E}\left[e^{h\overline{Y_i}}\right] \\ &= \prod_{i=1}^{n} \mathbb{E}\left[e^{hY_i}\right].\end{aligned}$$

□

Lemma 6.10 (Correlated Chernoff bounds, [36])**.** *Let $X_1, \ldots, X_n$ be $\{0,1\}$-valued random variables, let $Y_i = 1 - X_i$ and let also $X = \sum_{i=1}^{n} X_i$. If $Y_1, \ldots, Y_n$ are negatively correlated, then for any $\delta \in (0,1]$ it follows:*

$$\mathbb{P}\left[X \leq (1-\delta)\mathbb{E}[X]\right] \leq \exp\left(-\delta^2 \frac{\mathbb{E}[X]}{2}\right).$$

Proof. Let $Y = \sum_{i=1}^{n} Y_i$, let $p_i = \mathbb{E}[Y_i]$ and let $p = \sum_{i=1}^{n} \frac{p_i}{n} = \frac{\mathbb{E}[Y]}{n}$. By definition of expectation, for any $h > 0$ and for every $i \in [n]$ we have $\mathbb{E}[e^{hY_i}] = 1 - p_i + p_i e^h$. We can therefore use the above claim and the arithmetic mean-geometric mean inequality to write:

$$\mathbb{E}\left[e^{hY}\right] \leq \prod_{i=1}^{n} (1 - p_i + p_i e^h) \leq (1 - p + pe^h)^n.$$

Now, observe that for any s positive real, by Markov inequality and the last centred equation, we can write:

$$\begin{aligned}\mathbb{P}[Y \geq s] &= \mathbb{P}[e^{hY} \geq e^{hs}] \\ &\leq e^{-hs}\mathbb{E}[e^{hY}] \\ &\leq e^{-hs}(1 - p + pe^h)^n.\end{aligned}$$

In particular, for $\alpha = \delta \frac{\mathbb{E}[X]}{n}$:

$$\begin{aligned}\mathbb{P}[X \leq \mathbb{E}[X] - \alpha n] &= \mathbb{P}[Y \geq (p+\alpha)n] \\ &\leq \left(e^{-(p+\alpha)h}(1 - p + pe^h)\right)^n.\end{aligned}$$

Setting h such that $e^h = \frac{(p+\alpha)(1-p)}{p(1-p-\alpha)}$, we obtain

$$\mathbb{P}[Y \geq (p+\alpha)n] \leq \left(\left(\frac{p}{p+\alpha}\right)^{p+\alpha} \left(\frac{1-p}{1-p-\alpha}\right)^{1-p-\alpha}\right)^n.$$

Our result now follows from calculus. Indeed, if we set

$$f(x) = \ln\left(\left(\frac{p}{p+x}\right)^{p+x} \left(\frac{1-p}{1-p-x}\right)^{1-p-x}\right),$$

what we want to show is that $f\left(\frac{\delta\mathbb{E}[X]}{n}\right)n \le -\frac{\delta^2\mathbb{E}[X]}{2}$. In order to do so, we can notice that:

$$f'(x) = \ln\left(\frac{p(1-p-x)}{(p+x)(1-p)}\right),$$
$$f''(x) = -\frac{1}{(p+x)(1-(p+x))}.$$

Moreover, we have $f(0) = f'(0) = 0$. Now we can analyse the function $h(t) = f\left(t\frac{\mathbb{E}[X]}{n}\right)$ for $t \in [0,1]$. Firstly, note that $h''(t) = \frac{E[X]^2}{n^2}f''\left(t\frac{\mathbb{E}[X]}{n}\right)$; our final computation is as follows:

$$\begin{aligned}
h''(t) &= \frac{E[X]^2}{n^2}f''\left(t\frac{\mathbb{E}[X]}{n}\right)\\
&= -\frac{E[X]^2}{n^2}\frac{1}{\left(\frac{n-\mathbb{E}[X]}{n}+t\frac{\mathbb{E}[X]}{n}\right)\left(1-\frac{n-\mathbb{E}[X]}{n}-t\frac{\mathbb{E}[X]}{n}\right)}\\
&= -\frac{\mathbb{E}[X]^2}{\mathbb{E}[X](1-t)(n-(1-t)\mathbb{E}[X])}\\
&\le -\frac{E[X]}{n}.
\end{aligned}$$

Therefore, we can apply Taylor's theorem to obtain $f\left(\frac{\delta\mathbb{E}[X]}{n}\right) \le -\frac{\delta^2\mathbb{E}[X]}{2n}$, as we wanted. □

6.3 Outline of the proof

We first clarify some notation and terminology.

Notation. Let G be a simple graph. With partial k-colouring of G we mean a map $c : V(G) \to [k] \cup \{\text{'blank'}\}$ such that if $uv \in E$ and also $c(v) \in [k]$, then $c(v) \neq c(u)$.
Let c be a partial k-colouring of G, we denote with B_c the set of blank vertices of G; that is,

$$B_c = \{v \in V(G) : c(v) = \text{'blank'}\}.$$

Let c be a partial k-colouring of G and $v \in B_c$, we denote with $\mathcal{L}(c)_v$ the list of possible colours for v. More formally,

$$\mathcal{L}(c)_v = \{i \in [k] : \forall u \in N_G(v),\ c(u) \neq i\}.$$

Let us denote the partial list $\mathcal{L}(c)$ associated with c as $\mathcal{L}(c) = (\mathcal{L}(c)_v)_{v \in B_c}$.

If c' is an $\mathcal{L}(c)$-acceptable colouring of $G[B_c]$, then the unique k-colouring which is naturally denoted by $c \cup c'$ is a k-colouring of G (we mean the colouring C such that $C\big|_{B_c} = c'$ and $C\big|_{V(G)\setminus B_c} = c$).

Notation. Let G be a simple graph, let $\mathcal{L}$ be a list assignment for the vertices of G with colours from $[k]$, and let $i \in [k]$ be a colour. For $v \in V(G)$, we use the following notation to indicate the vertices adjacent to v with i in their list assignment,

$$T^{\mathcal{L}}_{v,i} = \{u \in N_G(v) : i \in \mathcal{L}_u\}.$$

For $\ell \in \mathbb{N}^+$, we say that a partial assignment of colours $\mathcal{L}$ is ℓ-good, if $|\mathcal{L}_v| \geq \ell$ for every vertex v, and if for every choice of $v \in V(G)$ and $i \in \mathcal{L}_v$ we have $\left|T^{\mathcal{L}}_{v,i}\right| \leq \frac{\ell}{8}$.

We are now ready to start following the outline of the proof of Theorem 6.1. We first need two lemmas.

Lemma 6.11. *Let G be a simple graph and $\mathcal{L}$ an assignment of colours for its vertices with colours from $[k]$. If $\mathcal{L}$ is ℓ-good for some $\ell \in \mathbb{N}^+$, then there exists an $\mathcal{L}$-acceptable colouring of G. In particular, G is then k-colourable.*

Proof. Without loss of generality, we may assume that $|\mathcal{L}_v| = \ell$ for any $v \in V(G)$. Indeed, if this is not the case, we can take a list assignment $\overline{\mathcal{L}}$ such that $\left|\overline{\mathcal{L}}_v\right| = \ell$ and $\overline{\mathcal{L}}_v \subseteq \mathcal{L}_v$ for every $v \in V(G)$; if we prove that G is $\overline{\mathcal{L}}$-colourable, it follows that G is also $\mathcal{L}$-colourable.
Let c be a random colouring of the vertices of G obtained by choosing independently uniformly at random the colour of each vertex v among the colours of $\mathcal{L}_v$. Moreover, for any adjacent couple of vertices u, v, and for any $i \in \mathcal{L}_u \cap \mathcal{L}_v$, let us denote with $A_{uv,i}$ the event that both u and v are mapped to i by the random colouring c. We have, for u, v, i as above, $\mathbb{P}[A_{uv,i}] = \frac{1}{|\mathcal{L}_v||\mathcal{L}_u|} = \frac{1}{\ell^2}$ because by hypothesis c colours the vertices uniformly and independently, and because $|\mathcal{L}_v| = |\mathcal{L}_u| = \ell$. Finally, for sake of simplicity, let us denote with $\overline{A}$ the event in which c is an $\mathcal{L}$-acceptable colouring; more specifically:

$$\overline{A} = \bigcap_{(uv,i) \in \bigcup_{uv \in E(G)} \{uv\} \times (\mathcal{L}_u \cap \mathcal{L}_v)} \overline{A}_{uv,i}.$$

We want to show that $\mathbb{P}\left[\overline{A}\right] > 0$.
If $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are disjoint subsets of $V(G)$, then for any adequate $i, j \in [k]$ we have that $A_{u_1v_1,i}$ and $A_{u_2v_2,j}$ are mutually independent because of the mutual independence principle (for a reference [31, Chapter 4]). Therefore, each $A_{uv,i}$ can be dependent of at most $\sum_{h \in \mathcal{L}_v} \left|T^{\mathcal{L}}_{v,h}\right| + \sum_{j \in \mathcal{L}_u} \left|T^{\mathcal{L}}_{u,j}\right| \leq \ell\frac{\ell}{8} + \ell\frac{\ell}{8} = \frac{\ell^2}{4}$ other events. Hence, we can apply the Lopsided Local Lemma 6.6 to show that the probability of $\overline{A}$ is strictly positive.
Let Γ be the simple graph over the set $\bigcup_{uv \in E(G)} \{uv\} \times (\mathcal{L}_u \cap \mathcal{L}_v) \subseteq E(G) \times [k]$ with (u_1v_1, i) and (u_2v_2, j) being an edge if and only if $\{u_1, v_1\}$ and $\{u_2, v_2\}$ have a common element. Because of what we said in the last paragraph, this is a lopsidependency graph for the events $A_{uv,i}$ (it follows by the independence of non-adjacent events).
Then, in order to apply the Lopsided Local Lemma, we just need to observe that by multiplying the maximum degree of Γ, which is at most $\frac{\ell^2}{4}$, with the upper bound for the probability of any $A_{uv,i}$, which is $\frac{1}{\ell^2}$, we obtain: $\frac{\ell^2}{4}\frac{1}{\ell^2} = \frac{1}{4}$. Therefore proving by Lemma 6.6 that $\mathbb{P}\left[\overline{A}\right] > 0$. For any c such that $\overline{A}$ occurs, we notice that c is an $\mathcal{L}$-acceptable colouring. □

The following lemma is the last essential step we need to prove Theorem 6.1.

Lemma 6.12. *For every positive real ε, there exists $\Delta_\varepsilon \in \mathbb{N}$ such that any triangle-free graph G with maximum degree $\Delta \geq \Delta_\varepsilon$ has a partial k-colouring c such that $\mathcal{L}(c)$ is ℓ-good for $k \leq (1+\varepsilon)\frac{\Delta}{\ln(\Delta)}$ and $\ell = \lfloor \Delta^{\frac{\varepsilon}{2}} \rfloor$.*

Theorem 6.1 follows from this last result.

Proof of Theorem 6.1. Using the notation of Lemma 6.12, we can obtain a partial k-colouring of G such that $\mathcal{L}(c)$ is ℓ-good, for $k \leq (1+\varepsilon)\frac{\Delta}{\ln(\Delta)}$. We can use Lemma 6.11 to conclude that there exists a k-colouring of G. And therefore $\chi(G) \leq k \leq (1+\varepsilon)\frac{\Delta}{\ln(\Delta)}$. □

It remains to show that Lemma 6.12 holds.

6.4 The last piece

In order to obtain the desired result, we first need a technical lemma, that allows us to work locally with our colourings.
By uniform random partial $\mathcal{L}(c)$-acceptable k-colouring of a set of vertices $N \subseteq B_c$ we mean an element taken uniformly at random in the set of $\mathcal{L}(c)$-acceptable $[k] \cup \{\text{‘blank’}\}$-colourings of $G[N]$.

Lemma 6.13. *For every ε positive real, there exists $\Delta_\varepsilon \in \mathbb{N}$ such that, if G is a triangle-free graph of maximum degree $\Delta \geq \Delta_\varepsilon$, if we set $k = \lfloor (1+\varepsilon)\frac{\Delta}{\ln(\Delta)} \rfloor$ and $\ell = \lfloor \Delta^{\frac{\varepsilon}{2}} \rfloor$ we have the following. Let c_0 be a fixed partial k-colouring of G and $u \in V(G)$ such that $\{u\} \cup N_G(u) \subseteq B_{c_0}$. Let c' be a uniformly random partial $\mathcal{L}(c_0)$-acceptable k-colouring of $N_G(u)$. Then we have $\mathbb{P}[|\mathcal{L}(c')_u| < \ell] \leq \frac{\Delta^{-3}}{8}$; moreover, $\mathbb{P}\left[\exists i \in [k] \text{ s.t. } i \in \mathcal{L}(c')_u \text{ and } \left|T_{u,i}^{\mathcal{L}(c')}\right| > \ell/8\right] \leq \frac{\Delta^{-3}}{8}$.*

Proof. Since G is triangle-free and $N_G(u)$ is an independent set, an appropriate model for c' is as follows: the partial colouring c' associates to each vertex $v \in N_G(u)$ a uniform random element of $\mathcal{L}(c_0)_v \cup \{\text{‘blank’}\}$; this is done independently for each vertex. For $i \in [k]$, we may notice that $i \in \mathcal{L}(c')_u$ if and only if c' does not assign i to any vertex in $N_G(u)$ (and hence, if and only if c' does not assign i to any vertex in $T_{u,i}^{\mathcal{L}(c_0)}$). By our model, for $v \in T_{u,i}^{\mathcal{L}(c_0)}$, it is clear that c' is equally likely to assign any of $|\mathcal{L}(c_0)_v| + 1$ possible distinct values to v; therefore we can write, for any $i \in [k]$,

$$\begin{aligned}
\mathbb{P}[i \in \mathcal{L}(c')_u] &= \prod_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \left(1 - \frac{1}{|\mathcal{L}(c_0)_v| + 1}\right) \\
&\geq \prod_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \exp\left(-\frac{1}{|\mathcal{L}(c_0)_v|}\right) \\
&= \exp\left(-\sum_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \frac{1}{|\mathcal{L}(c_0)_v|}\right).
\end{aligned}$$

The first equality holds because, in order to compute the probability, we are only interested in the colouring of the vertices adjacent to u which might be coloured with i by c'; therefore we are only interested in the value that c' attains in $T_{u,i}^{\mathcal{L}(c_0)}$. For each of these vertices independently, the probability that c' assigns them the colour i is exactly $\frac{1}{|\mathcal{L}(c_0)_v|+1}$ because of our model for c'. The inequality follows from calculus.

Let us denote with r_i the value $r_i = -\sum_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \frac{1}{|\mathcal{L}(c_0)_v|}$. We have the following bound:

$$\begin{aligned} -\sum_{i\in[k]} r_i &= \sum_{i\in[k]} \sum_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \frac{1}{|\mathcal{L}(c_0)_v|} \\ &= \sum_{v\in N_G(u),\ \mathcal{L}(c_0)_v \neq \emptyset} \frac{1}{|\mathcal{L}(c_0)_v|} \sum_{i\in[k]} \mathbb{1}_{i\in\mathcal{L}(c_0)_v} \\ &= |\{v \in N_G(u) : \mathcal{L}(c_0)_v \neq \emptyset\}| \\ &\leq \Delta. \end{aligned}$$

We can use this upper bound to find a lower bound on the expectation of $|\mathcal{L}(c')_u|$; indeed, by convexity of the exponential function:

$$\begin{aligned} \mathbb{E}[|\mathcal{L}(c')_u|] = \mathbb{E}\left[\sum_{i\in[k]} \mathbb{1}_{i\in\mathcal{L}(c')_u}\right] &= \sum_{i\in[k]} \mathbb{P}[i \in \mathcal{L}(c')_u] \\ &\geq \sum_{i\in[k]} \exp(r_i) \geq k \exp\left(\frac{1}{k}\sum_{i\in[k]} r_i\right) \\ &\geq k\exp\left(-\frac{\Delta}{k}\right) \geq (1+\varepsilon)\frac{\Delta}{\ln(\Delta)} \exp\left(-\frac{\ln(\Delta)}{1+\varepsilon}\right) \\ &= (1+\varepsilon)\frac{\Delta^{\frac{\varepsilon}{1+\varepsilon}}}{\ln(\Delta)} \geq 2\ell. \end{aligned}$$

The inequality in the last line hods for large enough Δ. We want to use the Chernoff bound as shown in Lemma 6.10. In order to do so, we need to prove that the indicator variables $1 - \mathbb{1}_{i\in\mathcal{L}(c')_u}$ are negatively correlated. Recalling the definition of negative correlated random variables, we need to show that for any nonempty set $S \subseteq [k]$ it holds

$$\mathbb{P}[\forall i \in S,\ i \notin \mathcal{L}(c')_u] \leq \prod_{i\in S} \mathbb{P}[i \notin \mathcal{L}(c')_u].$$

Instead, we show that for any $i \in [k]$ and for any $S' \subseteq [k] \setminus \{i\}$ we have $\mathbb{P}\left[i \notin \mathcal{L}(c')_u | S' \cap \mathcal{L}(c')_u = \emptyset\right] \leq \mathbb{P}[i \notin \mathcal{L}(c')_u]$. This is sufficient; indeed, if we index S as $S = \{i_1, \ldots, i_t\}$, then we have:

$$\begin{aligned} \mathbb{P}[\forall i \in S,\ i \notin \mathcal{L}(c')_u] &= \prod_{j=1}^{t} \mathbb{P}\left[i_j \notin \mathcal{L}(c')_u | \{i_1, \ldots, i_{j-1}\} \cap \mathcal{L}(c')_u = \emptyset\right] \\ &\leq \prod_{i\in S} \mathbb{P}[i \notin \mathcal{L}(c')_u]. \end{aligned}$$

For fixed i and S' as above, we have that $\mathbb{P}\left[i \notin \mathcal{L}(c')_u | S' \cap \mathcal{L}(c')_u = \emptyset\right] \leq \mathbb{P}[i \notin \mathcal{L}(c')_u]$ is equivalent to $\mathbb{P}\left[S' \cap \mathcal{L}(c')_u = \emptyset | i \in \mathcal{L}(c')_u\right] \geq \mathbb{P}[S' \cap \mathcal{L}(c')_u = \emptyset]$ because of Remark 6.8.
This last centred inequality holds true. Indeed, the conditional probability in the LHS can be obtained by choosing the c' image colours of the vertices of $N_G(u)$ uniformly at random excluding i. In particular, for $v \in N_G(u)$, we

colour v uniformly and independently from the other vertices with a colour from $\mathcal{L}(c_0)_v \setminus \{i\}$. It is therefore true that for any $v \in N_G(u)$ and for any $j \in \mathcal{L}(c_0)_v \setminus \{i\}$, we can write

$$\mathbb{P}\left[c'(v) = j | i \in \mathcal{L}(c')_u\right] \geq \mathbb{P}\left[c'(v) = j\right].$$

And we recall that if $c'(v) = j$ then $j \notin \mathcal{L}(c')_u$. Therefore,

$$\mathbb{P}\left[S' \cap \mathcal{L}(c')_u = \emptyset | i \in \mathcal{L}(c')_u\right] \geq \mathbb{P}[S' \cap \mathcal{L}(c')_u = \emptyset]$$

as wanted.
Hence, by Chernoff Bound 6.10 we can write:

$$\begin{aligned}
\mathbb{P}[|\mathcal{L}(c')_u| \leq \ell] &\leq \mathbb{P}\left[|\mathcal{L}(c')_u| \leq \frac{1}{2}\mathbb{E}[|\mathcal{L}(c')_u|]\right] \\
&\leq \exp\left(-\frac{\mathbb{E}[|\mathcal{L}(c')_u|]}{8}\right) \\
&\leq \exp\left(-\frac{\ell}{4}\right) \leq \frac{\Delta^{-3}}{8}.
\end{aligned}$$

The last inequality holds for large enough Δ.
We now show that, with high probability, $\mathbb{P}\left[\left|T_{u,i}^{\mathcal{L}(c')}\right| > \ell/8\right] \leq \frac{\Delta^{-3}}{8}$ for every $i \in \mathcal{L}(c')_u$. Recall that, for $i \in [k]$, we set $r_i = -\sum_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \frac{1}{|\mathcal{L}(c_0)_v|}$; we now use for simplicity $\rho_i = -r_i$. Moreover, let

$$\Psi = \left\{i \in [k] : \rho_i > \Delta^{\frac{\varepsilon}{4}}\right\}.$$

To show that with high probability, there are no colours $i \in \mathcal{L}(c')_u$ with $\left|T_{u,i}^{\mathcal{L}(c')}\right| > \ell/8$ we work separately with the colours in Ψ and in $[k] \setminus \Psi$. Indeed, we show that with high probability $\Psi \cap \mathcal{L}(c')_u = \emptyset$, and that for $i \in [k] \setminus \Psi$ it holds with high probability $\left|T_{u,i}^{\mathcal{L}(c')}\right| \leq \ell/8$.

- We show that $\mathbb{P}[\Psi \cap \mathcal{L}(c')_u \neq \emptyset] \leq \frac{1}{2}\Delta^{-4}$ for high enough Δ;

$$\begin{aligned}
\mathbb{P}[\Psi \cap \mathcal{L}(c')_u \neq \emptyset] &\leq \mathbb{E}\left[|\Psi \cap \mathcal{L}(c')_u|\right] = \sum_{i \in \Psi} \mathbb{P}[i \in \mathcal{L}(c')_u] \\
&= \sum_{i \in \Psi} \prod_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \mathbb{P}\left[c'(v) \neq i\right] \\
&= \sum_{i \in \Psi} \prod_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \left(1 - \frac{1}{1 + |\mathcal{L}(c_0)_v|}\right) \\
&< \sum_{i \in \Psi} \prod_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \exp\left(-\frac{1}{2\mathcal{L}(c_0)_u}\right) \\
&< \sum_{i \in \Psi} e^{-\frac{1}{2}\rho_i} < k e^{-\frac{\Delta^{\varepsilon/4}}{2}} < \frac{1}{2}\Delta^{-4}.
\end{aligned}$$

  Therefore, with high probability, we are interested just in the colours in $[k] \setminus \Psi$.

- It holds $v \in T_{u,i}^{\mathcal{L}(c')}$ if and only if $v \in T_{u,i}^{\mathcal{L}(c_0)}$ and $c'(v) = \text{`}blank'$. Therefore, for any $i \notin \Psi$, we have:

$$\mathbb{E}\left[\left|T_{u,i}^{\mathcal{L}(c')}\right|\right] = \sum_{v \in T_{u,i}^{\mathcal{L}(c_0)}} \frac{1}{|\mathcal{L}(c_0)_v| + 1} < \rho_i \leq \Delta^{\varepsilon/4}.$$

  Because $N_G(u)$ is an independent set, and because by our model of c', the choices of whether a $v \in T_{u,i}^{\mathcal{L}(c_0)}$ is in $T_{u,i}^{\mathcal{L}(c_0)}$ (whether $c'(v) = \text{`}blank'$) are made independently for each v, standard concentration bounds apply. Therefore it holds (for a reference, McDiarmid [28, Theorem 2.3(b)]) for $\delta > 0$:

$$\mathbb{P}\left[\left|T_{u,i}^{\mathcal{L}(c')}\right| > (1+\delta)\mathbb{E}\left[\left|T_{u,i}^{\mathcal{L}(c')}\right|\right]\right] < \exp\left(-\delta^2 \frac{\mathbb{E}\left[\left|T_{u,i}^{\mathcal{L}(c')}\right|\right]}{2(1+\frac{\delta}{3})}\right).$$

  Which gives us that the probability that there is one $i \notin \Psi$ with $\left|T_{u,i}^{\mathcal{L}(c')}\right| > 2\Delta^{\frac{\varepsilon}{4}}$ is at most (for Δ large enough):

$$ke^{-\frac{3}{8}\Delta^{\varepsilon/4}} < \frac{1}{2}\Delta^{-4}.$$

Therefore, for large enough Δ, the probability that there exists $i \in \mathcal{L}(c')_u$ with $\left|T_{u,i}^{\mathcal{L}(c')}\right| > \frac{\ell}{8}$ is less than $\frac{\Delta^{-3}}{8}$. □

We need one last result in order to be able to prove Lemma 6.12.

Proposition 6.14. *Let c_0 be any fixed partial k-colouring of G, let $U \subseteq B_{c_0}$ and let c' be any fixed partial $\mathcal{L}(c_0)$-acceptable colouring of U. Let c be a uniformly chosen partial k-colouring of G. Then, if we denote $W = V(G) \setminus U$, it holds:*

$$\mathbb{P}\left[c\big|_U = c' \Big| c\big|_W = c_0\right] = \frac{1}{M},$$

where M is the number of partial $\mathcal{L}(c_0)$-acceptable colourings of U.

Proof. There are two possible cases:

- If $c\big|_U$ is not $\mathcal{L}(c_0)$-acceptable, then $\mathbb{P}\left[c\big|_W = c_0\right] = 0$, because $c_0 \cup c'$ would not be a partial k-colouring (and we know that c is).
- If $c\big|_U$ is a partial $\mathcal{L}(c_0)$-acceptable colouring, then

$$\begin{aligned}\mathbb{P}\left[c\big|_U = c' \Big| c\big|_W = c_0\right] &= \frac{\mathbb{P}[c = c_0 \cup c']}{\mathbb{P}\left[c\big|_W = c_0\right]} \\ &= \frac{1}{\#\mathcal{L}(c_0)\text{-accept. col. of } G}.\end{aligned}$$

  The last inequality holds because both the numerator and the denominator are independent from c'; therefore, every $\mathcal{L}(c_0)$-acceptable c' has the same probability of being equal to $c\big|_U$.

□

We are finally ready to prove our main lemma.

Proof of Lemma 6.12. Let c be chosen uniformly at random in the set of all partial k-colourings of G. For every $u \in V(G)$ we define the following bad event,

$$A_u = \left\{ u \in B_c \text{ and } \left(|\mathcal{L}(c)_u| < \ell \text{ or } \left| T_{u,i}^{\mathcal{L}(c)} \right| > \frac{\ell}{8} \text{ for some } i \in \mathcal{L}(c)_u \right) \right\}.$$

We prove that, with positive probability, none of these bad events A_u happens, and therefore with positive probability $\mathcal{L}(c)$ is ℓ-good.
We now build a lopsidependency graph Γ for the events A_u; we set Γ over $V(G)$ as follows: we add an edge uv in $E(\Gamma)$ if u and v have distance at most 3 in G. Therefore $\Delta(\Gamma) \leq \Delta^3 =: d$. In order to apply Lemma 6.6 we need to show that for $u \in V(G)$ and $S \subseteq V(G) \setminus (N_\Gamma(u) \cup \{u\})$ we have $\mathbb{P}\left[A_u \middle| \bigcap_{v \in S} \overline{A}_v\right] \leq p = \frac{\Delta^{-3}}{4}$. In this way, we obtain $4pd \leq 1$ and hence we can apply the Lopsided Local Lemma 6.6.
In the remaining of the proof, we denote by $N_G^2(v)$ the set of vertices with distance at most 2 from v in G. For any $v \in V(G)$, the event A_v is determined by $c|_{N_G^2(v)}$.
Fix a vertex $u \in V(G)$ and $S \subseteq V(G) \setminus N_\Gamma(u)$; by last paragraph, we have that $\bigcap_{v \in S} \overline{A}_v$ is determined by $c|_{V(G) \setminus N_G(u)}$. In particular, this means that for every partial colouring c_0 of $V(G) \setminus N_G(u)$ we either have that $\bigcap_{v \in S} \overline{A}_v$ holds or not; hence the partition of events in which each set is induced by a partial colouring of $V(G) \setminus N_G(u)$ behaves well with respect to the conditional probability of $\bigcap_{v \in S} \overline{A}_v$. Therefore, by the law of total probability 6.7, it suffices to show that, for every partial colouring c_0 of $V(G) \setminus N_G(u)$, we can write:

$$\mathbb{P}\left[A_u \middle| c|_{V(G) \setminus N_G(u)} = c_0\right] \leq \frac{\Delta^{-3}}{4}.$$

To this end, let c_0 be a partial colouring of $V(G) \setminus N_G(u)$. Without loss of generality, we may assume that $u \in B_{c_0}$, because otherwise the probability of A_u is null.
Now, by Proposition 6.14, $c|_{N_G(u)}$ conditioned on $c|_{V(G) \setminus N_G(u)} = c_0$ is a uniform random partial $\mathcal{L}(c_0)$-acceptable colouring of $N_G(u)$.
We are now in the same situation as the one in Lemma 6.13, therefore we can conclude that

$$\mathbb{P}\left[A_u \middle| c|_{V(G) \setminus N_G(u)} = c_0\right] \leq \frac{\Delta^{-3}}{4},$$

as needed. □

7 List colourability of non-uniform hypergraphs

In this final section, we present a generalisation of a result due to Duraj et al. [11]. This is another extremal example for non-2-colourable structures; the study of these objects relate to the Pentagon Conjecture, as we saw in Section 4.

7.1 A short history of the problem

One problem that has inspired a lot of research in Combinatorics is the one of finding the smallest value $m(k)$ such that there exists a k-uniform hypergraph with $m(k)$ edges that is not 2-colourable. This problem was formulated by Erdős and Hajnal [14] in the 60s. While the best known upper bound for $m(k)$, due to Erdős, is dated back to 1964 [13] and states $m(k) = O(k^2) \cdot 2^k$; the best lower bound, due to Radhakrishnan and Srinivasan [37], is more recent and gives $m(k) = \Omega((k/\ln k)^{1/2}) \cdot 2^k$. Moreover, thanks to Lovász [25] it is known that deciding whether a k-uniform hypergraph is 2-colourable is a NP-complete problem for $k \geq 3$.
An immediate generalisation to the above problem asks to find, for $r \in \mathbb{N}^{\geq 2}$ the minimum number $m(k,r)$ for which there exists a k-uniform hypergraph with $m(k,r)$ edges that is not r-colourable. This generalised formulation of the problem is due to Herzog and Schönheim [18] in the early 70s; in the same paper where they formulate this problem, they also provide the upper bound $m(k,r) \leq (1 + O(\frac{1}{k})\frac{e}{2}k^2(\ln(r))r^k$, which remains the best known. As in the case for $r = 2$, the lower bound had more success; the currently best is due to Kostochka [20].

Theorem 7.1 (Kostochka, [20])**.** *Let $r < \left(\frac{1}{8}\ln(\ln(k)/2)\right)^{1/2}$ then for $a = \lfloor \log_2(r) \rfloor$ it holds:*

$$m(k,r) > e^{-4r^2}\left(\frac{k}{\ln(k)}\right)^{\frac{a}{a+1}} r^k.$$

Another direction in which this problem can be generalised is by relaxing the uniformity condition (this reformulation is due to Erdős [12]). Let H be a k-uniform hypergraph and suppose that we colour each vertex of $V(H)$ independently, with a colour taken uniformly at random among r colours. It is immediate to notice that the expected number of monochromatic edges is

$$q(H) = \sum_{e \in E} r^{1-|e|}.$$

Because $q(H) = |E| \cdot r^{1-k}$ when H is uniform, finding $m(k,r)$ is equivalent to find a k-uniform hypergraph that is not r-colourable with minimal value of $q(H)$. But the definition of $q(H)$ does not require the k-uniformity, hence this last formulation allows us to generalise the problem to non-uniform hypergraphs. The case for $r = 2$ of this generalised problem has been studied by Beck, Shabanov et al. The best known result is as follows.

Theorem 7.2 (Duraj, Gutowski and Kozik, [11])**.** *There exists a constant $\delta > 0$ such that any hypergraph $H = (V,E)$ with $k = \min\{|e| : e \in E\}$ and $q(H) \leq \delta \cdot \ln(k)$ is 2-colourable.*

In this section, we provide a generalisation of this result to list-colourings.

7.2 Preliminaries

For any r positive integer, and for $H = (V, E)$ a hypergraph, an r*-list* for H is a list of allowed colours $\mathcal{L} = (\mathcal{L}_v)_{v\in V}$ such that for any $v \in V$ it holds $|\mathcal{L}_v| = r$. If for any given r-list $\mathcal{L} = (\mathcal{L}_v)_{v\in V}$ we have that H is $\mathcal{L}$-colourable, we say that H is r*-list colourable.*

It is practical to denote with $s_{\min}(H) = \min\{|e| : e \in E\}$; and, for $j \geq s_{\min}(H)$, we define $E_j = \{e \in E : |e| = j\}$; finally, we denote $q_j(r, H) := \sum_{e\in E_j} r^{1-j}$ and we let $q(r, H) := \sum_{j\geq s_{\min}(H)} q_j(H)$.

We prove the following result.

Theorem 7.3. *For every $r \in \mathbb{N}^{\geq 2}$, there exists $\delta = \delta(r) > 0$ such that if $H = (V, E)$ is an hypergraph with $k = s_{\min}(H)$ and $q(r, H) \leq \delta \frac{\ln(k)}{\ln\ln(k)}$ then H is r-list-colourable.*

We now fix $r \in \mathbb{N}^{\geq 2}$ for the rest of this section. We write $q = q(H) = q(H, r)$ and similarly for q_j.

Remark 7.4. Notice that every H with $q(H) \leq 1$ is r-list-colourable (the probability that a random uniform colouring is proper is positive); therefore in what follows we can assume without loss of generality that $q(H) > 1$. Also, for some fixed but yet to be decided $N = N(r) \in \mathbb{N}$, we can focus only on hypergraphs H with $s_{\min}(H) \geq N$. Indeed, by setting $\delta < \frac{1}{N}$ we obtain that only hypergraphs H with $s_{\min}(H) \geq N$ satisfy the hypothesis.

Therefore, from now until the end of this section, we fix a hypergraph $H = (V, E)$ with $k = s_{\min}(H) \geq N$ and $1 < q \leq \delta \frac{\ln(k)}{\ln\ln(k)}$ for some δ yet to be determined (but at most $\frac{1}{N}$, where N is yet to be determined too). Moreover we, fix an r-list of allowed colours $\mathcal{L} = (\mathcal{L}_v)_{v\in V}$. We want to prove that H is $\mathcal{L}$-colourable.

7.3 Definition of the random colouring

In this subsection, we define a random $\mathcal{L}$-colouring `AL` of H, and we show that `AL` is a proper colouring with positive probability.

Let $\mathscr{L} = \bigcup_{v\in V} \mathcal{L}_v$, and let $\prec$ be a cyclic total order on $\mathscr{L}$ (arbitrary, but fixed from now on). For $v \in V$ and $c \in \mathcal{L}_v$, let us also denote $P_v(c)$ the previous element of c in $\mathcal{L}_v$ with respect to $\prec$; similarly, let $S_v(c)$ be the next element of c in $\mathcal{L}_v$ with respect to $\prec$.

The random colouring `AL` is defined in two parts as follows.

`part 1` *Random colouring and weight.* Let $(C_v)_{v\in V}$ be a sequence of independent random variables, in which C_v has image space $\mathcal{L}_v$ and uniform distrubution (we call this colouring the *initial colouring* C, sometimes we refer to $(C_v)_{v\in A}$ with C_A for some set A). Let $(W_v)_{v\in V}$ be a sequence of independent random variables (also independent on the sequence $(C_v)_{v\in V}$), in which W_v is taken uniformly at random in $[0, 1]$ (we refer to these values as the *weights* of the vertices, and we also may use the notation W_A for some set $A \subset V$).

`part 2` *Recolouring.* After `part 1`, we define `AL` as follows. Let $V = \{v_1, \ldots, v_n\}$ in such a way that for every i we have $W_{v_i} \leq W_{v_{i+1}}$ (if there are choices to make, let them be taken uniformly at random; we may notice that if the W_v are all distinct, the procedure is completely deterministic). We define

`AL` one vertex of V one after the other, starting with v_1 and proceeding until v_n.

If v_i is the last vertex of a C-monochromatic edge e (meaning that we already defined the colour `AL` on all the other vertices of e), and if $\mathtt{AL}(z) = C_z$ for all the vertices in $e \setminus v_i$, we define $\mathtt{AL}(v) = S_v(C_{v_i})$ and we say that e is a *reason to recolour* v_i. Else, we define $\mathtt{AL}(v_i) = C_{v_i}$.

Remark 7.5. If all the weights in the sequence $(W_v)_{v \in V}$ are distinct, `part 2` is deterministic (the random choices are done in `part 1`).

We now define some events. Let $\alpha = r^2 \cdot 5000$ and $\varepsilon = \frac{1}{1000}$.

z) *Two vertices have the same weight.* Let $\mathcal{Z}$ be the event that there exist $v, w \in V$ such that $W_v = W_w$. We have $\mathbb{P}[\mathcal{Z}] = 0$ because the Lebesgue measure is non-atomic.

a) *Too many initially monochromatic edges.* The expected number of C-monochromatic edges is at most q. This follows by linearity of expectation and noticing that for every edge e the probability that e is monochromatic is at most $r^{1-|e|}$ (the value depends on $\mathcal{L}$). Let $\mathcal{A}$ be the event that there are more than αq monochromatic edges. By Markov's inequality, $\mathbb{P}[\mathcal{A}] < \frac{1}{\alpha}$.

b) *A light monochromatic edge.* Let us denote $p_j := \frac{\ln(\alpha q)}{j}$. There is $\delta_0 > 0$ not depending on H (but dependent on r) such that if $1 < q < \delta_0 \frac{\ln(k)}{\ln\ln(k)}$ then $p_j \in (0, 1)$. Let us assume $\delta < \delta_0$.

An edge $e \in E_j$ is called light if $W(e) := \max_{v \in e} W_v \leq 1 - p_j$. The expected number of light monochromatic edges of size j is at most:

$$q_j r^{j-1} \cdot (1 - p_j)^j \cdot r^{1-j} < \frac{q_j}{\alpha q},$$

where the first term $q_j r^{j-1}$ equals $|E_j|$. If we fix an edge $e \in E_j$ the term $(1 - p_j)^j$ is the probability that every vertex in e has weight between 0 and $1 - p_j$; while the last vertex is an upper bound on the probability that e is monochromatic.

Therefore, the expected number of light monochromatic edges is at most $\frac{1}{\alpha}$. Let $\mathcal{B}$ be the event that there is a light monochromatic edge; then $\mathbb{P}[\mathcal{B}] \leq \frac{1}{\alpha}$.

c) *Too many almost monochromatic edges.* Let Q_j be the random variable that counts the number of almost monochromatic edges of size j (an almost monochromatic edge is an edge in which all vertices have the same colour, except for at most one vertex). Since the number of possible almost monochromatic edges cannot exceed the number of certifying pairs associated with edges of size j (ordered pairs (e, v) with e of size j and $v \in e$), we have that:

$$\mathbb{E}[Q_j] \leq q_j r^{j-1} \cdot j \cdot r^{2-j} = r \cdot j \cdot q_j,$$

where the first two terms $q_j r^{j-1} \cdot j = |E_j| j$ account for the number of possible certifying pairs associated with edges of size j; while r^{2-j} is the probability that, given a certifying pair (e, v), it holds that $e \setminus v$ is monochromatic.

We define $Y := \sum_j \frac{Q_j}{j}$; from the above upper bound, we obtain $\mathbb{E}[Y] \leq rq$. Let $\mathcal{C}$ denote the event that $Y > \alpha q$; by Markov's inequality, $\mathbb{P}[\mathcal{C}] < \frac{r}{\alpha}$.

Let $\mathcal{G}$ be the event $\mathcal{G} = \neg(\mathcal{Z} \vee \mathcal{A} \vee \mathcal{B} \vee \mathcal{C})$. We have $\mathbb{P}[\mathcal{G}] \geq 1 - \varepsilon$.

7.4 Analysis of the colouring

We start with a technical lemma, then we proceed.

Lemma 7.6. *Let X be a nonnegative random variable, bounded above by M; suppose moreover that $\mathbb{E}[X] \leq \lambda M$. Then, for any convex function $f : [0, M] \to [0, \infty)$ with $f(M) \geq f(0)$, the following inequality holds:*

$$\mathbb{E}[f(X)] \leq \lambda f(M) + (1 - \lambda) f(0).$$

It is important to notice the following.

Remark 7.7. Remember that our goal is to prove that with positive probability $\texttt{AL}$ is a proper $\mathcal{L}$-colouring of H (where H is the hypergraph we fixed and $\mathcal{L}$ is the arbitrary r-list for H that we choose). Let $\mathcal{M}_{\texttt{AL}}(e)$ be the event that the edge e is $\texttt{AL}$-monochromatic. We have:

$$\begin{aligned}
\mathbb{P}[\texttt{AL} \text{ proper}] &= 1 - \mathbb{P}[\exists e : \mathcal{M}_{\texttt{AL}}(e)] \\
&= 1 - (\mathbb{P}[\mathcal{G} \cap \exists e : \mathcal{M}_{\texttt{AL}}(e)] + \mathbb{P}[\overline{\mathcal{G}} \cap \exists e : \mathcal{M}_{\texttt{AL}}(e)]) \\
&\geq 1 - (\mathbb{P}[\mathcal{G} \cap \exists e : \mathcal{M}_{\texttt{AL}}(e)] + \mathbb{P}[\overline{\mathcal{G}}]) \\
&\geq 1 - \varepsilon - \sum_{e \in E} \mathbb{P}[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}(e)].
\end{aligned}$$

Let us fix an arbitrary edge e. Because each vertex of e has a list of size r, it holds that e can be monochromatic of at most r distinct colours. Let us fix $\texttt{Red} \in \bigcap_{v \in e} \mathcal{L}_v$ arbitrarily and let $\mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)$ be the event that e is coloured $\texttt{Red}$-monochromatically by $\texttt{AL}$. We prove

$$\mathbb{P}[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)] \leq \frac{1}{(r+1) \cdot q \cdot r^{|e|-1}}.$$

This allows us to write:

$$\begin{aligned}
\mathbb{P}[\texttt{AL} \text{ proper}] &\geq 1 - \varepsilon - \sum_{e \in E} \sum_{\texttt{Red} \in \bigcap_{v \in e} \mathcal{L}_v} \mathbb{P}[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)] \\
&\geq 1 - \varepsilon - \sum_{e \in E} r \cdot \frac{r^{1-|e|}}{(r+1) \cdot q} = 1 - \varepsilon - \frac{r}{r+1} > 0.
\end{aligned}$$

Therefore, Theorem 7.3 follows from the following lemma.

Lemma 7.8. *Let $e \in E$ and* $\texttt{Red} \in \bigcap_{v \in C} \mathcal{L}_v$. *Then*

$$\mathbb{P}[\mathcal{G} \cap \mathcal{M}_{AL}^{\texttt{Red}}(e)] \leq \frac{1}{(r+1) \cdot q \cdot r^{|e|-1}}.$$

Let us now fix for the rest of this section $e \in E$ of size s and $\texttt{Red} \in \bigcap_{v\in V} \mathcal{L}_v$ (if this set is empty the above lemma holds trivially).

Remark 7.9. Almost surely, AL is a deterministic function that has as input some random variables. In particular, we have $\texttt{AL} = \texttt{AL}((C_v)_{v\in V}, (W_v)_{v\in V})$. We may notice that:

i) Let $\mathcal{M}_C^{\texttt{Red}}$ be the event that e is C-monochromatic. Then $\mathbb{P}[\mathcal{M}_C^{\texttt{Red}} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}] = 0$. Indeed, let $((C_v)_{v\in V}, (W_v)_{v\in V})$ be in $\overline{\mathcal{Z}} \cap \mathcal{M}_C^{\texttt{Red}}$. Then `part 2` of the AL algorithm recolours at least one of the vertices of e; therefore $((C_v)_{v\in V}, (W_v)_{v\in V})$ is not in $\mathcal{M}_{\texttt{AL}}^{\texttt{Red}}$.

ii) Let f be any edge in E and $v \in f$; we denote with $\mathcal{H}_{f,v}$ the event that f is the reason to recolour v. Let $f \in E$ with $|e \cap f| > 1$. Then $\mathbb{P}[(\bigcup_{v\in e\cap f} \mathcal{H}_{f,v}) \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)] = 0$. Indeed, suppose that f is a reason to recolour $v \in e \cap f$; then v is the heaviest vertex of f and all the other vertices of f were not recoloured during `part 2` (in particular, there is another vertex w in $f \cap e$ for which $\texttt{AL}(w) = C(w) \neq \texttt{AL}(v)$). Therefore the final colouring of e is not monochromatic. This means that if AL colours e monochromatically, and $|f \cap e| > 1$, then f is not the reason to recolour v for any v in the intersection (if we are in $\mathcal{G}$).

iii) For any $((C_v)_{v\in V}, (W_v)_{v\in V}) \in \mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)$ there is at least 1 and at most αq vertices in e with initial colour different from Red. Indeed, in `part 2`, there are at most many recoloured vertices as many initially monochromatic edges, and because $((C_v)_{v\in V}, (W_v)_{v\in V}) \in \overline{\mathcal{A}}$ we have that there are at most αq initially monochromatic edges.

iv) Let us fix for this paragraph a vector $C'_{V\setminus e} = ((C'_v)_{v\in V\setminus e})$ (which represents a partial colouring of V); for any edge $f \neq e$ such that $|f \cap e| = 1$, let us say that f endangers the vertex $v = f \cap e$ if $f \setminus e$ is $P_v(\texttt{Red})$-monochromatic with respect to $C'_{V\setminus e}$. We call severity of v the minimum size of f such that f endangeres v, and we denote with $\mathcal{R}_j^e$ the set of vertices of e that are endgangered with severity j; we also set $\left|\mathcal{R}_j^e\right| = R_j^e$. It is important to notice that if $((C_v)_{v\in V}, (W_v)_{v\in V}) \in \mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)$ is such that $C_{V\setminus e} = C'_{V\setminus e}$, then every C-non-Red vertex w in e is endangered, and C-coloured with $P_w(\texttt{Red})$. Therefore, if the severity of such a w is j, then $W_w \geq 1 - p_j$ because the vector is in $\overline{\mathcal{B}}$ and there is a C-monochromatic edge of size j with heaviest vertex w.

The next step in our proof of Lemma 7.8 is as follows.
We may notice that $\mathcal{R}_j^e$ depends only on $C_{V\setminus e}$; hence, we define the random variable $X = \sum_{j\geq k} \left|\mathcal{R}_j^e\right| \cdot p_j$ over the probability space of the uniform colourings of $V \setminus e$.

Proposition 7.10. *Let x be a value such that $\mathbb{P}[X = x] > 0$. Then,*

$$\mathbb{P}\left[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e) \middle| X = x\right] < \frac{e^x - 1}{r^{|e|}}.$$

Proof. Let us fix a vector $C'_{V\setminus e} = ((C'_v)_{v\in V\setminus e})$ as before. We want to compute

$\mathbb{P}\left[\mathcal{G}\cap\mathcal{M}^{\texttt{Red}}_{\texttt{AL}}(e)\middle|C_{V\setminus e}=C'_{V\setminus e}\right]$, i.e. the probability of $\mathcal{G}\cap\mathcal{M}^{\texttt{Red}}_{\texttt{AL}}(e)$, when we impose the value of the colouring in $V\setminus e$ (notice that X, as R^e_j, is completely determined by $C_{V\setminus e}$). We have the following bound:

$$\mathbb{P}\left[\mathcal{G}\cap\mathcal{M}^{\texttt{Red}}_{\texttt{AL}}(e)\middle|C_{V\setminus e}=C'_{V\setminus e}\right]\leq$$
$$\leq\frac{1}{r^{|e|-\sum_j R^e_j}}\sum_{1\leq c_k+\ldots\leq\alpha q}\prod_j\binom{R^e_j}{c_j}\left(\frac{p_j}{r}\right)^{c_j}\left(\frac{1}{r}\right)^{R^e_j-c_j}.$$

The RHS term is explained as follows:

- The random variables C_e are independent from $C_{V\setminus e}$; also observe that, if $(C_V,W_v)\in\mathcal{G}\cap\mathcal{M}^{\texttt{Red}}_{\texttt{AL}}(e)$ we must have that all the non endangered vertices in e are coloured `Red`. This happens with probability $\frac{1}{r^{|e|-\sum_j R^e_j}}$ in the conditioned probability space (as in the original space, because of independence), which accounts for the first term in the RHS.
- The sum represents a partition. We are partitioning the event $(C_V,W_V)\in\mathcal{G}\cap\mathcal{M}^{\texttt{Red}}_{\texttt{AL}}(e)$ and we want to study the conditional probability of each of the partitioning sets. Each of the partitioning set is determined by the choice of which of the endangered vertices is coloured `Red` by C and which vertex v is coloured $P_v(\texttt{Red})$. In the above RHS, c_j corresponds to the number of initially non-`Red` vertices in $\mathcal{R}^e_j$. In each partitioning events, $c_k,c_{k+1},\ldots$ is fixed. Once the number of non-`Red` elements in each $\mathcal{R}^e_j$ is fixed, we have $\binom{R^e_j}{c_j}$ possible ways in which to choose the vertices that have to be non-`Red` (and therefore they have to be coloured $P_v(\texttt{Red})$) among the endangered vertices.
- At this point, every vector in our partitioning event has the same colouring (because we selected the vectors with a specific pattern of `Red` vertices in e, and the colouring of $V\setminus e$ is determined). Moreover, each of these vectors in $\mathcal{G}\cap\mathcal{M}^{\texttt{Red}}_{\texttt{AL}}(e)$ has to be such that each of the non-`Red` vertices in R^e_j has weight at least $1-p_j$. The probability of this partitioning event is exactly $\left(\frac{p_j}{r}\right)^{c_j}\left(\frac{1}{r}\right)^{R^e_j-c_j}$. Indeed, all the random variables are independent, the probability that one vertex is coloured by exactly one fixed colour is $\frac{1}{r}$ (this colour may be either `Red` or $P_v(\texttt{Red})$), and the probability that a specific vertex has weight at least $1-p_j$ is p_j (for any vertex it is the same).

Because in the inequality above we are just using the value of R^e_j we can write:

$$\mathbb{P}\left[\mathcal{G}\cap\mathcal{M}^{\texttt{Red}}_{\texttt{AL}}(e)\middle|(R^e_j)_j\right]\leq\frac{1}{r^s}\sum_{c=1}^{\alpha q}\sum_{c_k+\ldots=c}\prod_j\binom{R^e_j}{c_j}p_j^{c_j}$$
$$\leq\frac{1}{r^s}\sum_{c=1}^{\alpha q}\frac{1}{c!}\left(\sum_j R^e_j\cdot p_j\right)^c.$$

The first inequality follows from calculus, while the second inequality is due to the multinomial theorem and the bound $\binom{a}{b}\leq\frac{a^b}{b!}$. We may also notice that the last expression can be written as a function of X. Therefore we obtain:

$$\mathbb{P}\left[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e) \middle| X = x\right] \leq \frac{1}{r^s} \sum_{c=1}^{\alpha q} \frac{x^c}{c!} < \frac{e^x - 1}{r^s}.$$

□

We now proceed with the proof of the main lemma.

Proof of lemma 7.8. Recalling the definition of R_j^e and Q_j, it is clear that it holds $R_j^e \leq Q_j$. This implies:

$$X = \sum_j R_j^e p_j \leq \ln(\alpha q) \cdot \sum_j \frac{Q_j}{j} = \ln(\alpha q) \cdot Y.$$

In particular, in every vector in $\mathcal{G}$, it holds $X \leq \ln(\alpha q)\alpha q$. Therefore we have:

$$\begin{aligned}
\mathbb{P}[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)] &= \sum_{x \in \mathbb{R}} \mathbb{P}\left[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e) \middle| X = x\right] \\
&= \sum_{x \leq \ln(\alpha q)\alpha q} \mathbb{P}[X = x] \cdot \mathbb{P}\left[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e) \middle| X = x\right] \\
&< \sum_{x \leq \ln(\alpha q)\alpha q} \mathbb{P}[X = x] \frac{e^x - 1}{r^s} \\
&= \frac{1}{r^s} \mathbb{E}\left[e^{X'} - 1\right],
\end{aligned}$$

where $X' = X \cdot \mathbb{1}_{\leq \ln(\alpha q)\alpha q}$. Using the fact that $X' \leq X \leq \ln(\alpha q)Y$ we obtain:

$$\mathbb{E}[X'] \leq \sum_j q_j p_j \leq \frac{q \ln(\alpha q)\alpha q}{k}.$$

Moreover, by Lemma 7.6 applied over X' with $\lambda = \frac{q \ln(\alpha q)\alpha q}{k}$ and with $M = \ln(\alpha q)\alpha q$,

$$\mathbb{E}[e^{X'} - 1] \leq \frac{\exp(\ln(\alpha q)\alpha q)}{\alpha k}.$$

Which implies:

$$\mathbb{P}[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)] < \frac{1}{r^s} \frac{\exp(\ln(\alpha q)\alpha q)}{\alpha k}.$$

Now let $\delta < \frac{1}{2\alpha}$ (which we can do because α does not depend on H). Because

$$q < \delta \frac{\ln(k)}{\ln\ln(k)},$$

for k large enough (as we said, we can assume this by asking $\delta < \frac{1}{N}$ for some fixed N not depending on H) we obtain $\ln(\alpha q) \leq \ln\ln(k)$, which gives

$$\frac{\exp(\ln(\alpha q)\alpha q)}{\alpha k} \leq \frac{1}{\alpha k} \exp\left(\delta \frac{\alpha \ln k}{\ln\ln k} \ln\ln k\right) \leq \frac{k^{-\frac{1}{2}}}{\alpha}.$$

For k large enough, this last term is less than $\frac{1}{r(r+1)q}$ which gives us

$$\mathbb{P}[\mathcal{G} \cap \mathcal{M}_{\texttt{AL}}^{\texttt{Red}}(e)] \leq \frac{1}{(r+1)qr^{s-1}}.$$

□

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